Market Driven Systems (FRTN20)

Exercise 7. Linear Programming and Model Predictive Control.

Last updated: April 2012

1.

a. Each scalar constraint inequality (corresponding to a row in $Ax \preceq b$) defines an admissible half plane. The admissible region is the intersection of these half planes. Exchanging the inequality for an equality we get the boundary of the half plane. The first line gives

$$x_1 + x_2 = 1 \Leftrightarrow x_2 = -x_1 + 1, \tag{1}$$

which can easily be drawn. Pick an arbitrary point which does not lie on the boundary line, e.g, $x_1 = x_2 = 0$. This point fulfills the original inequality and hence we know which half plane defined by the line, is admissible. The procedure can be repeated for the remaining four inequalities, yielding the admissible region shown in Figure 1.

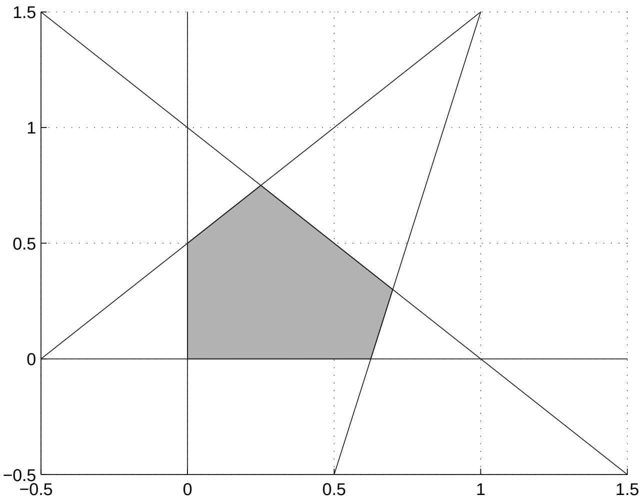

Figure 1

b. The vertices of the admissible region have the following x_1, x_2-coordinates:

$$v_0 = (0,0)\,,\; v_1 = \left(0, \frac{1}{2}\right),\; v_2 = \left(\frac{1}{4}, \frac{3}{4}\right),\; v_3 = \left(\frac{7}{10}, \frac{3}{10}\right),\; v_4 = \left(\frac{5}{8}, 0\right). \tag{2}$$

Evaluating the objective $f(x_1, x_2) = 5x_1 + 2x_2$ on the vertices gives

$$f(v_0) = 0,\; f(v_1) = 1,\; f(v_2) = \frac{11}{4},\; f(v_3) = \frac{41}{10},\; f(v_4) = \frac{25}{8}. \tag{3}$$

We conclude that the maximum occurs at v_3, corresponding to $x_1 = \frac{7}{10}$, $x_2 = \frac{3}{10}$.

Remark: Since the objective function is linear (a linear function is both convex and concave) and the admissible area is a convex polytope, the optimum must lie on a vertex of the polytope.

2.

a. The objective is to maximize the revenue $S_1x_1 + S_2x_2$.

b. Let $x = [x_1\ x_2]^T$. The following constraints must be fulfilled:

$$\begin{array}{ll} [1\ 1]\,x \preceq Q & \text{limit on total area} \\ [F_1\ F_2]\,x \leq F & \text{limit on fertilizer} \\ [P_1\ P_2]\,x \leq P & \text{limit on insecticide} \\ [-1\ 0]\,x \leq 0 & \text{non-negative wheat area} \\ [0\ -1]\,x \leq 0 & \text{non-negative barley area} \end{array}$$

c. The canonical form matrices can be obtained by simply stacking the constraints, e.g.

$$c = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix},\ A = \begin{bmatrix} 1 & 1 \\ F_1 & F_2 \\ P_1 & P_2 \end{bmatrix},\ b = \begin{bmatrix} Q \\ F \\ P \end{bmatrix}. \tag{4}$$

d. The admissible region is shown in Figure 2. See the solution of Problem 1 for details.

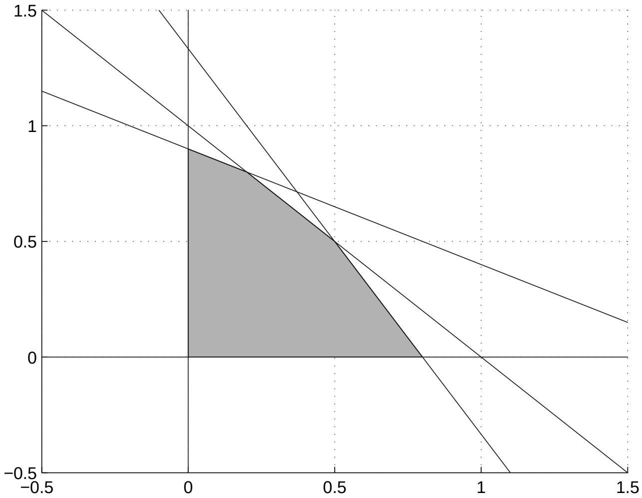

Figure 2

e. The vertices of the admissible region have the following x_1, x_2-coordinates:

$$v_0 = (0,0),\ v_1 = \left(0, \frac{9}{10}\right),\ v_2 = \left(\frac{1}{5}, \frac{4}{5}\right),\ v_3 = \left(\frac{1}{2}, \frac{1}{2}\right),\ v_4 = \left(\frac{4}{5}, 0\right). \tag{5}$$

Evaluating the objective $f(x_1,x_2) = 3x_1 + 2x_2$ on the vertices gives

$$f(v_0) = 0,\ f(v_1) = \frac{9}{5},\ f(v_2) = \frac{11}{5},\ f(v_3) = \frac{5}{2},\ f(v_4) = \frac{12}{5}. \tag{6}$$

We conclude that the maximum occurs at v_3, corresponding to $x_1 = x_2 = \frac{1}{2}$.

f. Removing the constraint on F is equivalent to removing v_2 and moving v_1 to v_1' with coordinates $x_1 = 0,\ x_2 = 1$. We observe that $f(v_1') = 2 < \frac{5}{2} = f(v_3)$. Hence F is not limiting the profit.

Removing the constraint on P corresponds to removing v_3 and moving v_4 to v_4' with coordinates $x_1 = 1,\ x_2 = 0$. We observe that $f(v_4') = 3 > \frac{5}{2} = f(v_3)$. Hence P is limiting the profit.

3. Mid-April corresponds to $t_{Apr} = \frac{3.5}{12}$, while mid-October occurs at $t_{Oct} = \frac{9.5}{12}$. The new objective functions f_{Apr} for mid-April and f_{Oct} for mid-October are given by

$$\begin{aligned} f_{Apr}(x_1, x_2) &= S_1^0 x_1 + S_2^0 \left(1 + \frac{1}{3}\sin(2\pi t_{Apr})\right) x_2 \\ f_{Oct}(x_1, x_2) &= S_1^0 x_1 + S_2^0 \left(1 + \frac{1}{3}\sin(2\pi t_{Oct})\right) x_2 \end{aligned} \tag{7}$$

The yearly revenue with the strategy from the previous problem is given by evaluating the objective functions at all vertices yields

$$\begin{aligned} &f_{Apr}(v_0) = 0,\ f_{Apr}(v_1) \approx 2.4,\ f_{Apr}(v_2) \approx 2.7,\ f_{Apr}(v_3) \approx 2.8,\ f_{Apr}(v_4) \approx 2.4 \\ &f_{Oct}(v_0) = 0,\ f_{Oct}(v_1) \approx 1.2,\ f_{Oct}(v_2) \approx 1.7,\ f_{Oct}(v_3) \approx 2.2,\ f_{Oct}(v_4) \approx 2.4. \end{aligned} \tag{8}$$

The old strategy turns out optimal for the April harvest, but not for the October harvest, where instead v_4 yields the maximal revenue. The yearly revenue when taking price variations into account is hence

$$f_{Apr}(v_3) + f_{Oct}(v_4) \approx 5.2, \tag{9}$$

which is an increase by 4 %.

For completeness, Figure 3 shows the vertex-vice revenue as a function of time. The vertices are:

Vertex	Line Style
v_0	not plotted
v_1	dash-dotted
v_2	dotted
v_3	solid
v_4	dashed

The figure shows that the optimum from the previous exercise is not optimal between mid-June and mid-Nov. (During this period, production corresponding to vertex v_4 is preferable.)

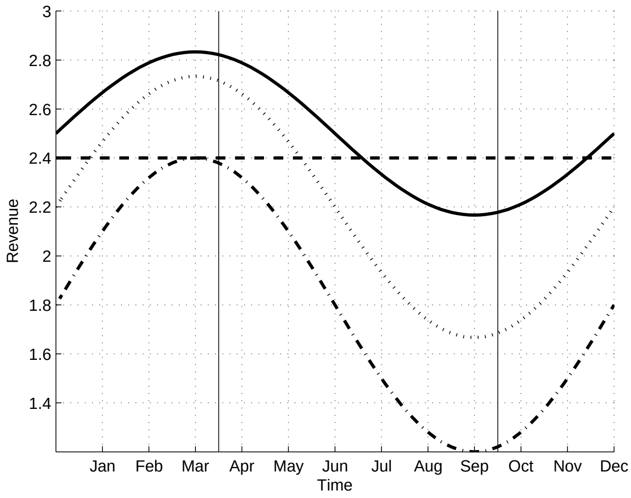
3
2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
Revenue
Jan
Feb
Mar
Apr
May
Jun
Jul
Aug
Sep
Oct
Nov
Dec
Time

Figure 3