Market Driven Systems (FRTN20)

Exercise 7. Linear Programming and Model Predictive Control.

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1. A common canonical form for linear programming is

$$\begin{aligned} \max_{x} \quad & c^T x \\ \text{subject to} \quad & Ax \preceq b,\ x \succeq 0. \end{aligned} \tag{1}$$

The optimization variable x is an $n \times 1$ real matrix. The $1 \times n$ real matrix c^T defines the optimization objective. The $m \times n$ matrix A and $m \times 1$ matrix b define hard constraints on x, that is they define a region in x-space where the optimum should be sought.

Consider the linear program (2) below.

$$\begin{aligned} \max_{x_1, x_2} \quad & \begin{bmatrix} 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \text{subject to} \quad & \begin{bmatrix} 1 & 1 \\ -2 & 2 \\ 8 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \preceq \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \succeq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned} \tag{2}$$

a. Determine and draw the admissible region in the x_1, x_2-plane defined by the constraints.

b. Solve the optimization problem and motivate why your method works.

2. A farmer has Q square kilometers of farm land, to be planted with either wheat or barley or some combination of the two. The farmer has a limited permissible amount F of fertilizer and P of insecticide, which can be used. The amount of fertilizer and insecticide required per square kilometer wheat are F_1 and P_1, respectively. The amounts required for barley are F_2 and P_2. The market prices for wheat and barley are S_1 and S_2, respectively. The area to be planted with wheat is x_1 while x_2 is the area to be planted with barley.

a. What is the optimization objective? Explain in your own words and write it down as a function of x. (Do not consider any constraints yet.)

b. What constraints are there? Explain them in words. Write each constraint on the form $A_k x \leq b_k$ for some matrices A_k and b_k.

c. Write the problem on the canonical form given by (1).

d. Draw the admissible x-region and when $S_1 = 3,\ S_2 = 2,\ Q = 1,\ F = 9,\ F_1 = 5,\ F_2 = 10,\ P = 4,\ P_1 = 5,\ P_2 = 3$.

e. Solve the optimization problem for the parameter set given in the previous sub-problem.

f. How would the optimum change if the restriction on fertilizer was removed? How would it change if the restriction on insecticide was removed? Which restriction is currently limiting the profit?

3. Consider again the farmer in the previous problem. It turns out that the grain market is subject to seasonal price variations due to export regulations. The dynamics of the variations are given by

$$\begin{aligned} S_1(t) &= S_1^0 \\ S_2(t) &= S_2^0 \left(1 + \frac{1}{3}\sin(2\pi t)\right), \end{aligned} \tag{3}$$

where t is the time in years and S_1^0, S_2^0 are the nominal prices from the previous exercise. I.e., the wheat price is constant, while barley is expensive in the spring and cheap during fall. The farmer lives in a region where grain can be harvested in April and October. How much can the farmer increase his revenue if he takes the price variation into account? Assume he sells all his harvest instantaneously.