Market Driven Systems (FRTN20)

Exercise 6

Last updated: April 2011

1.

a. The tilted line in the graph has the equation $1.5x - 425$. If we assume that $\varphi(x) \leq t$ we must have that $100 \leq t$ and $1.5x - 425$. Hence we get the minimization problem

$$\begin{aligned} \underset{x,t}{\text{minimize}}\ & t \\ \text{subject to: } & x \geq 0 \\ & x \leq 750 \\ & t \geq 100 \\ & t \geq 1.5x - 425 \end{aligned}$$

b. Let x_1 and x_2 denote how much of each product is to be produced, x_3, x_4 and x_5 be the usage of each saw and x_6 and x_7 the usage of the gluing machines.

The cost of using saw 1 will always be 200 and the usage is constrained by $x_3 \leq 500$. Similarly, the cost of saw 2 is 100 and $x_4 \leq 250$. The cost of saw 3 depends on the usage and will be $0.5x_5$. The usage of this saw is not constrained from above.

In the same way we understand that the cost of gluing machine 1 is $0.5x_6$ and the usage of this machine is constrained by $x_6 \leq 1000$. The way to formulate the cost of the second gluing machine was done in a. Note that we will change the variable name of t to x_8.

Hence the profit the company makes will be $21x_1 + 18x_2 - 200 - 100 - 0.5x_5 - 0.5x_6 - x_8$. Also, the total usage all the saws are used must greater than or equal to the needed usage to saw all material for both products, i.e. $7x_1 + 10x_2 \leq x_3 + x_4 + x_5$. The same must be true for the gluing, i.e. $16x_1 + 12x_2 \leq x_6 + x_7$. Hence we get the LP:

$$\begin{aligned} \underset{x}{\text{maximize}}\ & 21x_1 + 18x_2 - 200 - 100 - 0.5x_5 - 0.5x_6 - x_8 \\ \text{subject to: } & x \geq 0 \\ & 7x_1 + 10x_2 \leq x_3 + x_4 + x_5 \\ & 16x_1 + 12x_2 \leq x_6 + x_7 \\ & x_3 \leq 500 \\ & x_4 \leq 250 \\ & x_6 \leq 1000 \\ & x_7 \leq 750 \\ & 100 \leq x_8 \\ & 1.5x_7 - 425 \leq x_8 \end{aligned}$$

c. We easily understand that in optimality $x_8 = 100$. First of all we check that the proposed point actually is feasible, i.e. satisfies all the constraints. Now, if the point satisfies the KKT conditions (in both cases) it is indeed the optimal solution. The only active constraints are $7x_1 + 10x_2 \leq x_3 + x_4 + x_5$, $16x_1 + 12x_2 \leq x_6 + x_7$, $x_3 \leq 500$,

$x_4 \le 250$, $x_5 \ge 0$, $x_6 \le 1000$, $100 \le x_8$ and $1.5x_7 - 425 \le x_8$. Hence we only have 8 non-zero λ. The KKT condition (the gradient of the Lagrangian to be exact) becomes

$$\begin{aligned}
-21 + 7\lambda_1 + 16\lambda_2 &= 0 \\
-18 + 10\lambda_1 + 12\lambda_2 &= 0 \\
\lambda_3 - \lambda_1 &= 0 \\
\lambda_4 - \lambda_1 &= 0 \\
0.5 - \lambda_5 - \lambda_1 &= 0 \\
0.5 + \lambda_6 - \lambda_2 &= 0 \\
1.5\lambda_8 - \lambda_2 &= 0 \\
1 - \lambda_7 - \lambda_8 &= 0
\end{aligned}$$

which has the solution $\lambda = (\frac{9}{19}, \frac{21}{19}, \frac{9}{19}, \frac{9}{19}, \frac{1}{38}, \frac{23}{38}, \frac{5}{19}, \frac{14}{19})$. Since $\lambda \ge 0$ the proposed point is the optimal one.

d. If we examine the sawing constraint $7x_1 + 10x_2 - x_3 - x_4 - x_5 \le 0$, this means that the demanded sawing should be less than the available sawing. If the optimal dual variable λ_1^* associated to the constraint is greater than zero, by the condition that $(A_i x^* - b_i)\lambda_i^*$, then demanded sawing equals the available sawing. Hence, if we regard λ_1 as the price of not using the saw to its capacity, if that price is "right" we will use the sawing at its capacity. If the price is higher, then we would rather not use the sawing (and instead rent out the left-over capacity). If on the other hand the price is to low, the company would even like to pay for even more resources to do sawing.

2.

a. We only separate the constraints which includes variables of different "divisions". In this problem, those are the equality constraints. Introducing the prices λ_1 and λ_2, we get the optimization problem

$$\begin{aligned}
\underset{x}{\text{maximize}}\ & 21x_1 + 18x_2 - 200 - 100 - 0.5x_5 - 0.5x_6 - x_8 - \\
& \lambda_1(7x_1 + 10x_2 - x_3 - x_4 - x_5) - \lambda_2(16x_1 + 12x_2 - x_6 - x_7) \\
\text{subject to: }\ & x \ge 0 \\
& x_3 \le 500 \\
& x_4 \le 250 \\
& x_6 \le 1000 \\
& x_7 \le 750 \\
& 100 \le x_8 \\
& 1.5x_7 - 425 \le x_8
\end{aligned}$$

Now we can separate the problem

Production company:

$$\begin{aligned}
\underset{x_1, x_2}{\text{maximize}}\ & (21 - 7\lambda_1 - 16\lambda_2)x_1 + (18 - 10\lambda_1 - 12\lambda_2)x_2 \\
\text{subject to: }\ & x_1, x_2 \ge 0
\end{aligned}$$

Sawing companies:

$$\underset{x_3}{\text{maximize}}\ \lambda_1 x_3 - 200$$
$$\text{subject to: } 0 \leq x_3 \leq 500$$

$$\underset{x_4}{\text{maximize}}\ \lambda_1 x_4 - 100$$
$$\text{subject to: } 0 \leq x_4 \leq 250$$

$$\underset{x_5}{\text{maximize}}\ (\lambda_1 - 0.5)x_5$$
$$\text{subject to: } 0 \leq x_5$$

Gluing companies:

$$\underset{x_6}{\text{maximize}}\ (\lambda_2 - 0.5)x_6$$
$$\text{subject to: } 0 \leq x_6 \leq 1000$$

$$\begin{aligned} \underset{x_7,x_8}{\text{maximize}}\ & \lambda_2 x_7 - x_8 \\ \text{subject to: } & 0 \leq x_7 \leq 750 \\ & 100 \leq x_8 \\ & 1.5x_7 - 425 \leq x_8 \end{aligned}$$

b. If we examine the separated problem in the production company, we see that if the dual variables are not right, then the solution the minimization problem will be unbounded. Hence, if the production company where given the wrong dual variables, it would not be able to know how much it should produce. One way to come around this problem is described in the lecture slides. Here we introduce a form of averaging scheme so that x variables converges to the optimal value. One requirement of this aproach is that we must have a bounded region for allowed x. This means that we must choose a maximal allowed production of each product. The larger we choose this region, the slower the convergence will be to the optimal x.

Another way to handle the problem is to introduce a small concave perturbation, e.g. $-\varepsilon x^2$. The drawback of this approach is that we will actually change the optimal value a little bit.