Market-Driven Systems
Lecture 5

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Today's lecture

- Linear Programming (LP)
- LP in Production planning example
  - Static systems
  - Dynamical systems
- Model Predictive Control

Mini Problem

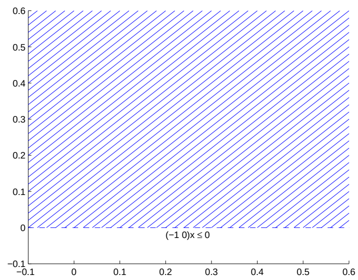
$$\begin{array}{ll} \text{Minimize} & -x_1 - x_2 \\ \text{subject to} & x_1 + 2x_2 \le 1 \\ & 2x_1 + x_2 \le 1 \\ & x_1 \ge 0 \\ & x_2 \ge 0 \end{array}$$

Equivalent matrix formulation:

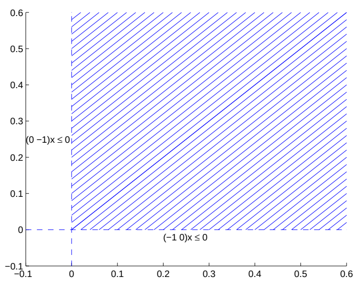
$$\begin{array}{ll} \text{Minimize} & \begin{pmatrix} -1 & -1 \end{pmatrix} x \\ \text{subject to} & \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} x \preceq \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ x \succeq 0 \end{array}$$

where $x = (x_1\ x_2)^T$

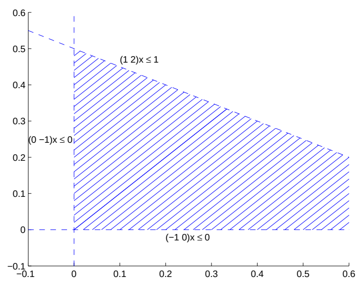
Mini Problem graphical solution



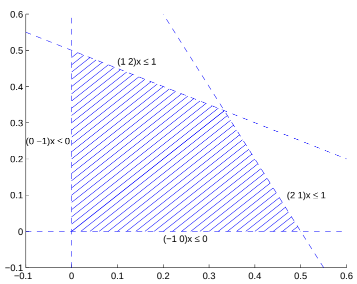
Mini Problem graphical solution



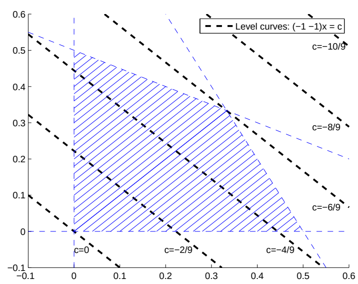
Mini Problem graphical solution



Mini Problem graphical solution



Mini Problem graphical solution



Linear Programming

General formulation:

$$\begin{array}{ll} \text{Minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & Hx = g \end{array}$$

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Production planning example

Two products are produced:

- Garden furniture
- Sleds

Two main parts of production

- Sawing
- Assembling

Production planning example cont'd

Weekly production:
x_1 : Garden furniture
x_2 : Sleds

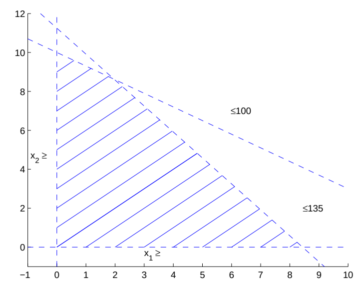
Product prices:
p_1 : Garden furniture
p_2 : Sleds

The objective is to maximize weekly profit:
$\max p_1 x_1 + p_2 x_2$

Subject to:
Sawing constraints: $7x_1 + 10x_2 \leq 100$
Assembling constraints: $16x_1 + 12x_2 \leq 135$

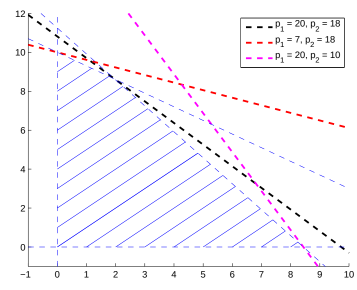
Production planning example cont'd

Sawing and assembling constraints:



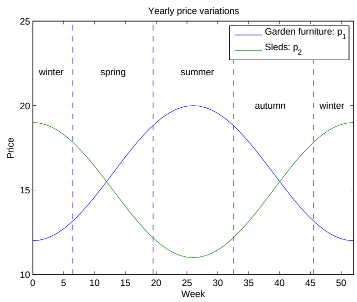
Production planning example cont'd

Level curves for optimal points obtained with different prices:



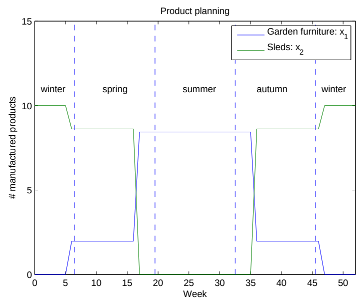
Production planning example cont'd

Seasonal variations in expected prices:



Production planning example cont'd

Optimal production for different seasons:



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Dynamic Production planning example

Hire extra personel to increase production:

Nominal learning (sawing):

$$x_3(t+1) = 0.7x_3(t) + 30u_3(t)$$

Nominal learning (assembling):

$$x_4(t+1) = 0.7x_4(t) + 40.5u_4(t)$$

where $u_3 \in [0, 1]$, $u_4 \in [0, 1]$ is fraction of full time employment

$x_3(t)$ and $x_4(t)$ quantifies increased capacity:

Sawing: $7x_1 + 10x_2 \leq 100 + x_3(t)$
Assembling: $16x_1 + 12x_2 \leq 135 + x_4(t)$

Mini problem

Assume that extra sawing personel is working full-time, i.e $u_3(t) = 1$, $t = 0, 1, \dots$

If the initial sawing capacity of the extra labor is 0, i.e $x_3(0) = 0$, what is the sawing capacity after three weeks, i.e. $x_3(3)$?

What is the stationary sawing capacity of the extra labor?

Mini problem - solution

Sawing capacity at time $t = 3$:

$$\begin{aligned} x_3(3) &= 0.7x_3(2) + 30u_3(2) = 0.7(0.7x_3(1) + 30u_3(1)) + 30u_3(2) \\ &= 0.7(0.7(0.7x_3(0) + 30u_3(0)) + 30u_3(1)) + 30u_3(2) \\ &= (0.7^2 + 0.7 + 1)30 = 65.7 \end{aligned}$$

Stationary capacity is given by:

$$x_3 = 0.7x_3 + 30$$

which gives

$$x_3 = \frac{30}{1 - 0.7} = \frac{30}{0.3} = 100$$

The total sawing capacity is doubled after learning period

Dynamic Production planning example cont'd

The weekly cost in hiring extra personel is p_3 and p_4 respectively

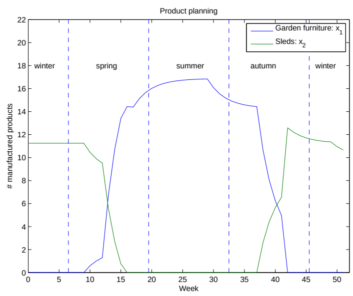
This gives the following production planning problem that optimizes one year ahead production:

$$\begin{array}{ll} \max & p_1(t)x_1(t) + p_2(t)x_2(t) - p_3(t)u_3(t) - p_4(t)u_4(t) \\ \text{subject to} & 7x_1(t) + 10x_2(t) \leq 100 + x_3(t) \\ & 16x_1(t) + 12x_2(t) \leq 135 + x_4(t) \\ & x_3(t+1) = 0.7x_3(t) + 30u_3(t) \\ & x_4(t+1) = 0.7x_4(t) + 40.5u_4(t) \\ & 0 \leq u_3(t) \leq 1 \\ & 0 \leq u_4(t) \leq 1 \\ & x_3(0) = x_3^0 \text{ , } x_4(0) = x_4^0 \end{array}$$

for $t = 0, ..., 52$ and x_3^0 and x_4^0 are the initial capacities for the extra personel

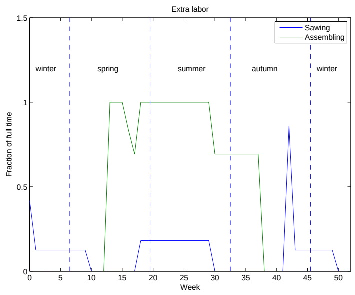
Dynamic Production planning example cont'd

Optimal production over 52 weeks with extra personel and product prices as before and $p_3 = p_4 = 100$:



Dynamic Production planning example cont'd

Optimal extra labor:



Dynamic Production planning example - limitations

The following is not compensated for:

- Prices may not be equal to predicted prices
- Extra personel might be fast or slow learners
- Decreased capacity due to employee illness
- ...

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Model Predictive Control (MPC)

Reoptimize every week to compensate for mis-match between reality and assumed model

Input to optimization problem:

- Current product prices, $p_1(t), p_2(t)$
- Current capacity of extra workers, $x_3(t), x_4(t)$

MPC Loop

MPC Loop; In the end of each week:

1. Get data for extra employment capacity and product prices for the past week, $(p_1(t), p_2(t), x_3(t), x_4(t))$
2. Solve the optimization problem with the gathered data as input
3. Hire extra personel the following week according to the obtained solution, $(u_3(t), u_4(t))$
4. In the end of next week repeat the procedure, i.e. Go to 1

MPC Example

Product planning example with model-reality mis-match:

Modeled employee learning:

$$x_3(t+1) = 0.7x_3(t) + 30u_3(t)$$
$$x_4(t+1) = 0.7x_4(t) + 40.5u_4(t)$$

Actual employee learning:

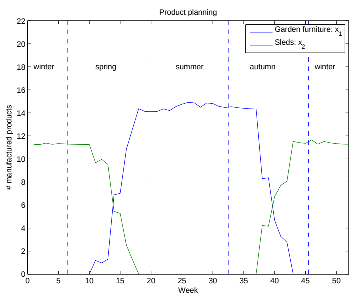
$$x_3(t+1) = 0.75x_3(t) + 30u_3(t) + v_3(t)$$
$$x_4(t+1) = 0.65x_4(t) + 40.5u_4(t) + v_4(t)$$

where $v_3(t)$ and $v_4(t)$ are uniformly distributed random numbers in $[-0.3x_3(t)\ 0]$ and $[-0.3x_4(t)\ 0]$ respectively

The product prices $p_1(t)$ and $p_2(t)$ are additively affected by uniformly distributed random noise in $[-1\ 1]$

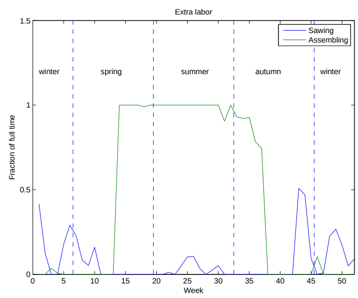
MPC Example - Results

Weekly production when extra labor decided using MPC:



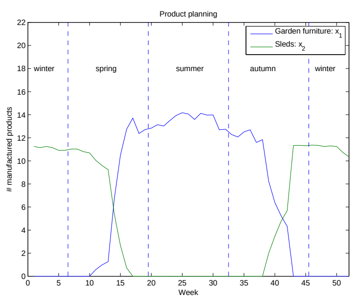
MPC Example - Results

Extra personel (decided using MPC):



MPC Example - Comparison

Production with extra labor as in dynamic production planning example (i.e. no feedback):



Profit over one year is 8.6% higher with MPC-feedback

MPC General Formulation

MPC introduced by an example. General formulation:

$$\text{Minimize} \quad \sum_{t=0}^{N} \ell(x(t), u(t))$$

$$\text{subject to} \quad x(t+1) = f(x(t), u(t)) \ , \ x(0) = x_0$$
$$x(t) \in X \ , \ u(t) \in U$$
$$\text{for } t = 0, \ldots, N$$

where x_0 is a measurement of the current state

MPC Loop - General

At time t:

1. Get measurements for $x(t)$, set $x_0 = x(t)$
2. Solve the optimization problem
3. Apply the optimzation result $u(0)$ to the system
4. After one sample, Go to 1 to repeat the procedure

MPC Pros and Cons

Pros:

- Good constraint handling
- Easily understandable tuning knobs (e.g. cost function)
- Usually gives good performance in practice
- Handles complex systems well

Cons:

- Calculation times
- System model needed
- Historically lack of theoretical understanding of the closed loop system

MPC Application

Pendulum - developed in our Department

MPC with quadratic cost and linear constraints

Summary

Model Predictive Control:

- Optimization based feedback of dynamical systems
- Explicit constraint handling
- Applicable to many types of problems