LUND UNIVERSITY

Department of
AUTOMATIC CONTROL

# FRTN 15 Predictive Control

**Final Exam March 14, 2018, 8am - 13pm**

## General Instructions

This is an open book exam. You may use any book you want, including the slides from the lecture, but no exercises, exams, or solution manuals are allowed. Solutions and answers to the problems should be well motivated. The exam consists of 6 problems. The credit for each problem is indicated in the problem. The total number of credits is 25 points. Preliminary grade limits:

| | |
|---|---|
| Grade 3: | 12 – 16 points |
| Grade 4: | 17 – 21 points |
| Grade 5: | 22 – 25 points |

## Results

The results of the exam will be presented in LADOK by March 21.

**1.** Let $p = d/dt$ and assume that we have a first order process, $y = G(p)u$, with

$$G(p) = \frac{b}{p+a},$$

for some *unknown* $a, b > 0$, and a reference model, $y_m = G_m(p)u_c$, with

$$G_m(p) = \frac{b_m}{p+a_m},$$

for some *known* parameter $a_m, b_m > 0$. Both she static gain and the pole location in the process model is unknown, leading us to consider a feedback

$$u(t) = \theta_1(t)u_c(t) - \theta_2(t)y(t).$$

With this feedback, we will choose the adaptive gains by the Lyapunov rule,

$$\frac{d\theta_1}{dt} \triangleq -\gamma u_c(t)e(t), \quad \frac{d\theta_2}{dt} \triangleq \gamma y(t)e(t). \tag{1}$$

for some positive constant $\gamma$.

**a.** With the proposed feedback law, write the transfer function from $u_c$ to $y$. Assume that the feedback gains, $\theta_1, \theta_2$, are constant and that process parameters are known. Which choice of gains result in perfect model-matching? (1 p)

**b.** As the process parameters are unknown, we will consider a MRAS driving the error $e = y - y_m$ for time-varying gains $\theta_1(t), \theta_2(t)$. Show that the function

$$\mathcal{V}(\mathbf{x}) = \frac{1}{2}\Big(e^2 + \alpha(b\theta_1(t) - b_m)^2 + \alpha(a + b\theta_2 - a_m)^2\Big),$$

is a valid Lyapunov function provided the Lyapunov update rule in (1). What could be a suitable state vector $\mathbf{x}$ in the Lyapunov function? (3 p)

*Solution*

**a.** The closed loop transfer function, $G_{cl}(p)$, from $u_c$ to $y$ is

$$y(t) = G_{cl}(p)u_c(t) = \frac{b\theta_1}{p+a+b\theta_2}u_c(t).$$

If $a, b$ are known, our goal is to achieve $G_{cl}(p) = G_m(p)$, whereby

$$\frac{b\theta_1}{p+a+b\theta_2} = \frac{b_m}{p+a_m} \rightarrow \begin{cases} \theta_1 = b_m/b \\ \theta_2 = (a_m - a)/b \end{cases}.$$

**b.** A valid Lyapunov function $\mathcal{V}(\mathbf{x})$ must satisfy

(1) Positive definiteness: $\mathcal{V}(\mathbf{x}) > 0$ for all $\mathbf{x}\backslash\{\mathbf{0}\}$,

(2) Radial unboundedness: $\|\mathcal{V}(\mathbf{x})\| \to \infty$ as $\|\mathbf{x}\| \to \infty$,

(3) Negative time derivative: $\dot{\mathcal{V}}(\mathbf{x}) \leq 0$.

The Lyapunov function is a positive definite function of some time-varying error $\mathbf{x}(t)$. When satisfying the above conditions, $\mathcal{V}(\mathbf{x})$ decreases in time, and our system is stable. If the inequality is replaced by negative definiteness, i.e. $\dot{\mathcal{V}}(\mathbf{x}) < 0$ for all $\mathbf{x}\backslash\{\mathbf{0}\}$, the the Lyapunov function will decrease until eventually reaching a minimum of $\mathcal{V}(\mathbf{x}) = 0$ at $\mathbf{x} = \mathbf{0}$, and our system is asymptotically stable.

From problem **a.**, the gains should be chosen such that

$$b\theta_1(t) \to b_m, \qquad a + b\theta_2(t) \to a_m,$$

for model matching. It is then reasonable to define an error-state

$$\mathbf{x}(t) = [x_1(t), x_2(t), x_3(t)]^T = [e(t), b\theta_1(t) - b_m, a + b\theta_2 - a_m].$$

whereby $\mathbf{x} = \mathbf{0}$ accomplishes model matching and a zero error $e(t)$. With this error-state, we set out to verify that

$$\mathcal{V}(\mathbf{x}) = \frac{1}{2}\Big(e^2 + \alpha(b\theta_1(t) - b_m)^2 + \alpha(a + b\theta_2 - a_m)^2\Big),$$

is a valid Lyapunov function. This function clearly satisfies the conditions (1) and (2) by definition. However, it remains to show that the time-derivative is decreasing. This is done by computing

$$\dot{\mathcal{V}}(\mathbf{x}) = e\dot{e} + \alpha(b\theta_1(t) - b_m)b\dot{\theta}_1 + \alpha(a + b\theta_2 - a_m)b\dot{\theta}_2, \tag{2}$$

as $\dot{x}_2 = b\dot{\theta}_1$ and $\dot{x}_3 = b\dot{\theta}_2$. The derivative $de/dt$ is found by first computing

$$\begin{aligned}
\frac{dy}{dt} &= -ay + bu = -ay + b(\theta_1(t)u_c(t) - \theta_2(t)y(t)),\\
\frac{dy_m}{dt} &= -a_m y_m + b_m u_c.
\end{aligned}$$

whereby

$$\begin{aligned}
\frac{de}{dt} &= \frac{dy}{dt} - \frac{dy_m}{dt}\\
&= -ay + b(\theta_1(t)u_c(t) - \theta_2(t)y(t) + a_m y_m - b_m u_c\\
&= -ay - b\theta_2(t)y(t) + a_m y_m + (b\theta_1(t) - b_m)u_c\\
&= -ay + a_m y - a_m y + a_m y_m - b\theta_2(t)y(t) + (b\theta_1(t) - b_m)u_c\\
&= -a_m e + (a_m - a - b\theta_2(t))y(t) + (b\theta_1(t) - b_m)u_c.
\end{aligned}$$

Insertion of the computed error time-derivative in (2) yields

$$\dot{\mathcal{V}}(\mathbf{x}) = -a_m e^2 + (b\theta_1(t) - b_m)(eu_c + \alpha b\dot{\theta}_1) + (a_m - a - b\theta_2(t))(ey - \alpha b\dot{\theta}_2),$$

We don't know what $b > 0$ is, but $\alpha > 0$ can be defined such that $\alpha b$ is any positive constant. Let $\alpha b \triangleq \gamma > 0$. Insertion of the Lyapunov-rule in (1) yields

$$\begin{aligned}
\dot{\mathcal{V}}(\mathbf{x}) =& -a_m e^2 + (b\theta_1(t) - b_m)(eu_c - eu_c) + (a_m - a - b\theta_2(t))(ey(t) - ey(t))\\
=& -a_m e^2 \leq 0,
\end{aligned}$$

as $a_m > 0$ by definition. We have then shown the conditions of (1) positive definiteness, (2) radial unboundedness and (3) a negative time-derivative.

**2.** Consider a double integrator system discretized at a time step of $h$ [s],

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} h^2/2 \\ h \end{bmatrix} u(k), \tag{3}$$

where $\mathbf{x}(k) = [\, x_1(k) \quad x_2(k) \,]^T$. This system is to follow some reference $\mathbf{x}_r(k) = [\, x_{1r}(k) \quad x_{2r}(k) \,]^T$ using an MPC controller to minimize a cost

$$J = \sum_{k=1}^{H_p} \|\mathbf{x}_r(t + hk) - \mathbf{x}(t + hk)\|_{\mathbf{Q}} + \sum_{k=1}^{H_u} \|\mathbf{u}_r(t + hk)\|_R,$$

for some $\mathbf{Q} \succ 0$, $R > 0$. The system is to follow a square positional reference in $x_{1r}(k)$, alternating between $\pm 1$ at a period of $T = 10$ [s], and a zero-velocity $x_{2r}(k) = 0$. For the combinations

(I) $\mathbf{Q} = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}, R = 1,$ (II) $\mathbf{Q} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}, R = 1,$

(III) $\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 100 \end{bmatrix}, R = 1,$ (IV) $\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, R = 100.$

the corresponding system response is shown in Figure 1.

**a.** Can the defined reference trajectory be followed by a double integrator? (1 p)

**b.** Pair each tuning (I)-(IV) with a plot (1)-(4) and give a short motivation. (1 p)

**c.** Assume that we intend on implementing the controller in practice, where we may expect a load disturbance on the control signal. Give an explanation as to why such disturbances may arise, and propose a modification of the linear model in (3) to achieve a better robustness against the load disturbance. (2 p)

*Solution*

**a.** The chosen reference trajectory cannot be followed by the system, as any change in position $x_1(t)$ requires a non-zero velocity $x_2(t)$. This is a common problem in control,

**b.** The correct answers are

(I)↔(1) As $Q_{11} >> Q_{22}, R$, we put a large cost on deviations $(x_{1r} - x_1)^2$ which results in aggressively following the positional reference trajectory.

(II)↔(2) As $Q_{11} = Q_{22} > R$, we put equal weight on positional and velocity reference tracking. We visibly see that $\int x_2^2 dt \approx \int (x_{1r} - x_1)^2 dt$.

(III)↔(3) As $Q_{22} >> Q_{11}, R$, we only care about following the velocity reference, which is zero, forcing the positional response to be close to constant.

(IV)↔(4) As $R >> Q_{11}, Q_{22}$, the control signal is kept very close to zero at all times, which results in very poor reference following in $x_1$ and $x_2$.

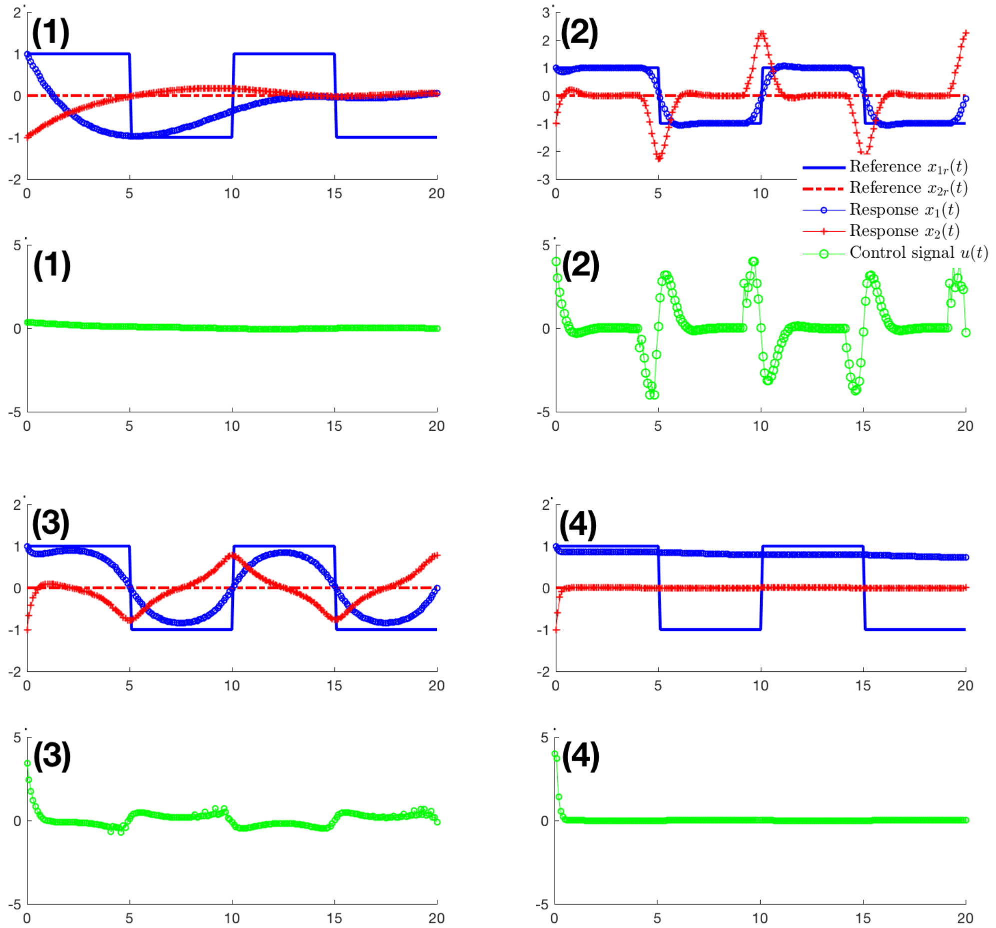


**Figure 1** Controlling the double integrator along a square reference in $x_{1r}(t)$ with $x_{2r}(t) = 0$. Responses in $x_1(t)$ (blue), $x_2(t)$ (red) and control signal $u(t)$ (green).

**c.** Reality is rarely linear, and assuming that the modelling errors are slowly time-varying, these may be modelled as a load disturbance. With a load disturbance on the control signal, $d(k)$, the dynamics may be written

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}(u(k) + d(k)),$$

where $\mathbf{A}$ and $\mathbf{B}$ are stated explicitly in (3). To incorporate an estimated disturbance $d(k)$ in the MPC formulation, we define an augmented state vector $\mathbf{x}^e(k) \triangleq [\mathbf{x}^T(k) \quad d(k)]^T$. The augmented system dynamics are then

$$\mathbf{x}^e(k+1) = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & 1 \end{bmatrix} \mathbf{x}^e(k) + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} u(k),$$

and to make use of this augmented model, we need to implement an estimator which estimates both the entire augmented state vector. As the system dynamics are linear, and assuming the noise distribution is gaussian, a Kalman filter may be a good choice.

**3.** Design a minimum-variance controller *with integral action* for the process

$$H_p(q) = \frac{b}{z-a},$$

subject to a load disturbance $v(t)$, such that

$$y(t+1) = ay(t) + b(u(t) + v(t)).$$

Answer the following questions.

**a.** Define a suitable disturbance model $A_d(z)v(t) = e(t+1)$. (1 p)

**b.** Using minimum degree pole placement design, find a negative feedback law $R(z)u(t) = -S(z)y(t)$ expressed in terms of the process parameters. (2 p)

**c.** What can you say about the feedback? Where are the closed loop poles located? Are minimum variance controllers generally robust to changes in the modelled noise characteristics? (1 p)

*Solution*

**a.** By the internal model principle, incorporating integral action is done by modelling an integral disturbance. A suitable noise model is given by

$$A_d(z)v(t) = v(t+1) - v(t) = e(t+1),$$

i.e., a discrete time integrator $z/(z-1)$.

**b.** With the disturbance model, the system is written

$$A_d(z)A(z)y(t) = b(A_d(z)u(t) + e(t+1)).$$

We write this as

$$A'y = B'u + C'\bar{e},$$

with

$$\begin{aligned} A' &= AA_d = (z-a)(z-1) \\ B' &= BA_d = b(z-1) \\ C' &= z^2 \\ \bar{e}(t) &= be(t-1) \end{aligned}$$

The Diophantine equation we have to solve is

$$A'R + B'S = z^2 B',$$

with the constraint that $B'$ is a factor of $R$. To get a monic $R(q)$ we divide the right hand side by $b$. This will give us a solution, $R(z)$ and $S(z)$, that is a factor $b$ smaller which does not affect the controller $-S(z)/R(z)$. We solve

$$A'R + B'S = z^2 B'/b,$$

with

$$R = R'B'/b.$$

Cancellation of $B'/b$ gives

$$(z-a)(z-1)R' + bS = z^2.$$

The minimum degree solution has $\deg R' = 0$ and $\deg S = 1$. We get $R' = 1$ and the coefficients in $S(z)$ can be computed from

$$\begin{aligned} z^1: &\qquad 0 = -a - 1 + bs_0 \\ z^0: &\qquad 0 = a + bs_1 \end{aligned}$$

This gives

$$\begin{aligned} R(z) &= z + r_1 = R'(z)B'(z)/b = z - 1 \\ S(z) &= s_0 z + s_1 = b^{-1}(a+1)z - b^{-1}a \end{aligned}$$

and the control law parametrised by $a$ and $b$ becomes

$$u(t) = \frac{S(z)}{R(z)}y(t) \Leftrightarrow u(t) = u(t-1) + \frac{1}{b}\Big((a+1)y(t) - ay(t-1)\Big).$$

**c.** With perfect knowledge of the process parameters, we get the controller

$$H_c(z) \triangleq \frac{S(z)}{R(z)} = \frac{b^{-1}(a+1)z - b^{-1}a}{z-1},$$

which, with negative feedback, results in a closed loop system

$$H_{cl}(z) = \frac{H_c(z)H_p(z)}{1 + H_c(z)H_p(z)} = \frac{(1+a)z - a}{z^2},$$

recognised as a deadbeat controller, as the poles of the characteristic polynomial are located in the origin - in effect, we have designed a deadbeat PI-controller. As we showed in the Laboratory 2 exercises, the minimum variance controller is very sensitive to changes in the modelled disturbance characteristics, and may perform worse than a standard PI controller if the noise is mischaracterised.

**4.** From optimal control, it is known that for a function $Q^* : x, u \to \mathbb{R}$ satisfying

$$Q^*(x_t, u_t) = c_t + \min_{u'} Q^*(x_{t+1}, u') \tag{4}$$

where $x_t, u_t, c_t$ are the state, control and cost at time $t$, the optimal state-feedback controller given by

$$u_t = \arg\min_{u'} Q^*(x_t, u'). \tag{5}$$

An adaptive controller is obtained by adaptive estimation of the function $Q$. To this end, we introduce $Q_w(x, u)$ as a linear combination of known functions $\phi : x, u \to \mathbb{R}$, parameterized by $w$, according to

$$Q_w(x, u) = \sum_i \phi_i(x, u)w_i = \Phi(x, u)^\intercal w, \quad w \in \mathbb{R}^I.$$

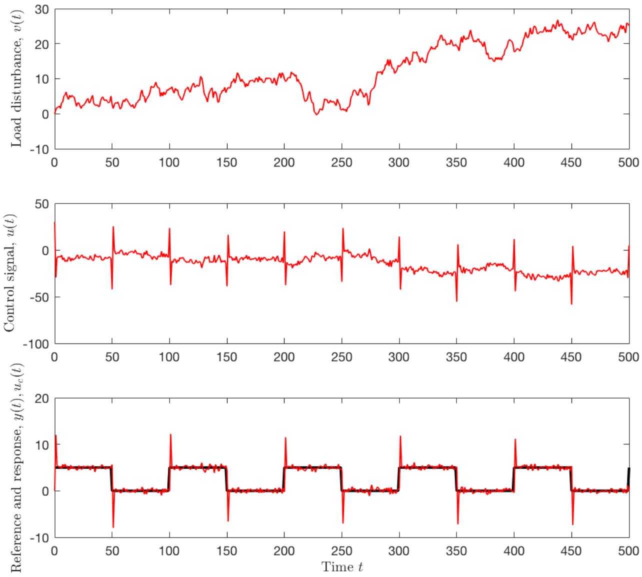


**Figure 2** System response with $e(t) \sim \mathcal{N}(0, 1)$, $a = 1.4$, $b = 0.4$ discretized at $h = 1$ with $v(t)$ (top) $u(t)$ (centre) and $y(t)$ combined with $u_c(t)$ (bottom).

The goal is thus to adapt the parameters $w$ such that our function $Q_w$ respects (4). To this end, consider at time $t$ the value produced by $\min_{u'} Q_w(x_{t+1}, u')$ to be fixed, such that $y_t = c_t + \min_{u'} Q_w(x_{t+1}, u')$ is a known scalar quantity given a trajectory $\{x_\tau, u_\tau, c_\tau\}_{\tau=1}^{t+1}$. The problem is now reduced to that of estimating $w$ such that $Q_w(x_t, u_t) = y_t$ and we want to do this by minimizing the following cost function

$$\sum_t \big(y_t - Q_w(x_t, u_t)\big)^{\mathsf{T}}\big(y_t - Q_w(x_t, u_t)\big). \tag{6}$$

**a.** Devise a method to minimize (6) with respect to $w$ online, i.e., the parameters $w$ are to be updated after each time-step. (2 p)

**b.** If we are able to estimate $Q$ accurately, (5) will be an optimal controller under the cost function that generates $c_t$. The optimization problem in (5) must be solved at each time step to calculate the control signal and this can be hard to do fast enough if the sample time is small. To circumvent this problem, we consider a new function, $\mu_p(x) : x \to \mathbb{R}$, parameterized by $p$, according to

$$u_t = \mu_p(x_t) = \sum_i \phi_i(x, u)p_i = \Phi(x, u)^{\mathsf{T}} p, \quad p \in \mathbb{R}^I,$$

and adjust the parameter vector $p$ online with the following update rule

$$p \leftarrow p - \alpha \Phi(x, u) \frac{d}{du} Q_w(x, u)|_{u=\mu(x_t)}, \tag{7}$$

where $\alpha$ is a small step-size parameter. Relate the update rule (7) to the optimization problem (5) and motivate why $u_t = \mu_p(x_t)$ will be a good approximation of (5). (2 p)

*Solution*

**a.** The problem of estimating $w$ in $Q_w(x_t, u_t) = \Phi(x,u)^\intercal w = y_t =$ is linear in the parameters and can thus be solved using, e.g., the recursive least-squares method for a quadratic cost function or gradient descent for a general (differentiable) cost function.

**b.** The update rule (7) is equal to

$$p \leftarrow p - \alpha \frac{d\mu}{dp} \frac{d}{du} Q_w(x,u)|_{u=\mu(x_t)},$$

which shows that the update rule performs gradient descent with respect to $p$ of $Q_w(x_t, \mu(x_t))$, hence, the parameters $p$ will be updated in the direction of decreasing $Q(x_t, \mu(x_t))$, which will on convergence lead to $\mu(x_t)$ being a good approximation to the solution of optimization problem (5).

**5.** Given the discrete-time system $Y(z) = G(z)U(z)$ with

$$G(z) = \frac{z+2}{z(z+0.5)}$$

we want the output, $y$, to follow a given reference signal $y_r$. In this exercise we assume $y_r$ is a step function, ie $y_r(t) = 1, \ \ t \geq 0$

**a.** The following ILC algorithm (with $Q = 1$)

$$u_{k+1}(t) = u_k(t) + \gamma q^n e_k(t)$$

can be used to find an input signal $u(t) = \lim_{k\to\infty} u_k(t)$ that solves the problem. Figure 3 illustrates the amplitude curves $|I - \gamma z^n G(z)|_{z=e^{j\omega}}$ for some different values of $\gamma$ and $n$. Also shown is the successful result of 100 iterations of the ILC algorithm for one of these choices of parameters, the other three parameter combinations do not work. Which parameter set is the working one (a,b,c or d)? Do not forget to motivate your answer. (1 p)

**b.** Calculating $U$ as

$$U(z) = G^{-1}(z)Y_r(z) = \frac{z(z+0.5)}{z+2} Y_r(z)$$

and using the inverse Z-transform to obtain $u$ does not work. Why? (1 p)

**c.** Describe how to calculate a bounded control signal, $u(t), \ -\infty < t < \infty$, that will make the output equal to a step function (without using ILC iteration). *Hint. Split the inverse of $G(z)$ into causal and anti-causal parts. Describe how to interpret the different parts as stable recursions to find u.* (1 p)

**d.** Look at the plot of the control signal resulting from the ILC procedure. Describe what the control signal is doing during the time interval before the step in the reference signal to be able to accurately track the step. (1 p)

*Solution*

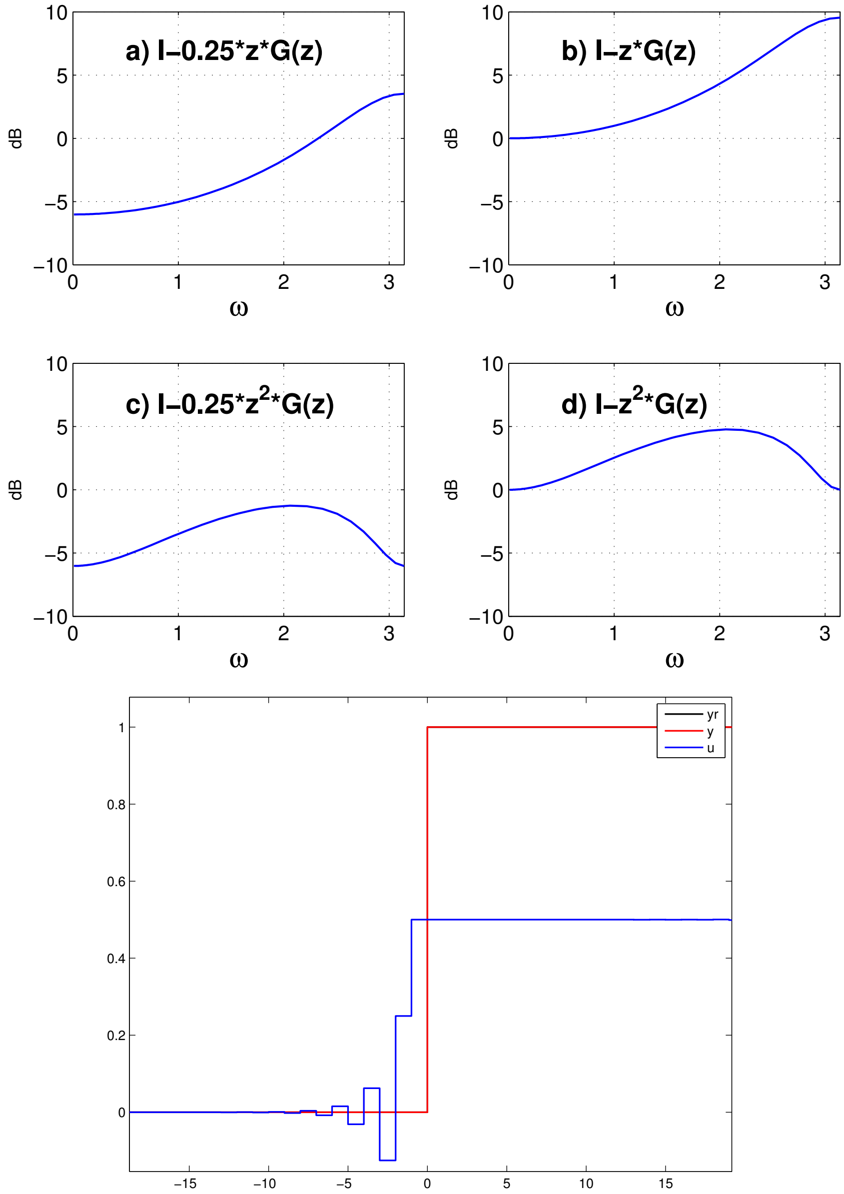


**Figure 3** The ILC amplitude curves a-d (top) for different combinations of $\gamma$ and $n$. The successful result after 100 iterations of the ILC algorithm (bottom).

**a.** The system depicted in c) is the only candidate with a maximum amplitude less than 1, resulting in stable ILC iterations.

**b.** The system can not simply be inverted because of the zero outside the unit circle, which upon inversion turns into an unstable pole. The calculated control signal would contain an unbounded term of the form $(-2)^k$.

**c.** An expansion of the inverse gives:

$$G^{-1}(z) = \frac{z(z+0.5)}{z+2} = z - 1.5 + \frac{3}{z+2}$$

This means that $u(t) = y_r(t+1) - 1.5y_r(t) + x(t)$ where

$$x(t+1) + 2x(t) = 3y_r(t).$$

This recursion can be rewritten as a stable anticausal filter

$$x(t) = -0.5x(t+1) + 1.5y_r(t).$$

The solutions to this with $y_r(t) = 1$ for $t \geq 0$ is $x(t) = (-2)^t$, when $t \leq 0$, and $x(t) = 1$ for $t \geq 0$. The solution is hence

$$u(t) = (-2)^t, t \leq -2, \quad u(t) = 0.5, t \geq -1.$$

which is very close to the control signal generated by the ILC shown in the figure.

**d.** The zero, located in $z = -2$ is located on the negative real axis, corresponding to an oscillation at the Nyquist frequency. Since the magnitude of the zero is greater than one, this oscillation is increasing in amplitude. When the system is subjected to an input with these properties, nothing will be visible in the output.

We see that the control signal is oscillating with increasing amplitude in advance of the step change in the reference. We can interpret this as the controller pumping energy into the system in such a way that the output remains at zero, i.e., exciting the zero dynamics of the system using the signal described above. When the step change occurs, the built up energy is released in such a way that the output makes exactly the desired jump.

**6.** Consider the system

$$\begin{aligned} x(k+1) &= x(k) + v(k) \\ y(k) &= x(k) + e(k) \end{aligned}$$

where $v \sim N(0, 1)$, $e \sim N(0, R_e)$ are uncorrelated with zero mean.

**a.** Describe the stationary Kalman filter that gives $\hat{x}(k \mid k-1)$ for the system. Is the performance better/worse/the same as that of an exponential filter

$$\hat{x}(k+1) = (1-\alpha)\hat{x}(k) + \alpha y(k)$$

with well chosen $\alpha$? (2 p)

**b.** Determine the scalar stationary covariance $P$ and the stationary gain $K = APC^T(R_e + CPC^T)^{-1}$ as functions of $R_e$. (1 p)

**c.** Now consider the case where the state is affected by a colored noise sequence $\hat{v}$, modeled as normal white noise $v$ passing through the system

$$\frac{C(z)}{A(z)}$$

where $C(z)$ and $A(z)$ are known polynomials. Explain how to implement an optimal Kalman filter in this case. (2 p)

*Solution*

**a.** The stationary Kalman filter without direct term is given by

$$\hat{x}_{k+1|k} = \hat{x}_{k|k-1} + K(y_k - \hat{x}_{k|k-1})$$

It is an exponential first order filter, the performance is hence the same.

**b.**

$$P = P + 1 - P^2(R_e + P)^{-1} \Rightarrow P = \frac{1}{2} + \sqrt{\frac{1}{4} + R_e} \Rightarrow P = \frac{1 + \sqrt{1 + 4R_e}}{2},$$
$$K = \frac{P}{R_e + P} = \frac{1}{R_e/P + 1} = \frac{1 + \sqrt{1 + 4R_e}}{2R_e + 1 + \sqrt{1 + 4R_e}},$$

$P$ is increasing and $K$ is decreasing as functions of $R_e$.

**c.** The state-space system must now be augmented with the noise model $C/A$

$$x(k+1) = \begin{bmatrix} 1 & C_v \\ \mathbf{0} & A_v \end{bmatrix} \begin{bmatrix} x \\ x_v \end{bmatrix} + \begin{bmatrix} 0 \\ B_v \end{bmatrix} v(k) \tag{8}$$

$$y(k) = \begin{bmatrix} 1 & \mathbf{0} \end{bmatrix} \begin{bmatrix} x \\ x_v \end{bmatrix} + e(k) \tag{9}$$

$A_v, B_v, C_v$ is a state-space realization of $C/A$ with $A_v$ having eigenvalues corresponding to the roots of the polynomial $A(z)$, the state vector $x_v$ contains the state of the noise model and $\mathbf{0}$ is a matrix of zeros with appropriate dimensions.