LUNDS TEKNISKA HÖGSKOLA
Lunds universitet

Institutionen för
REGLERTEKNIK

FRTN15 Predictive Control

Final Exam January 11, 2013, 8am - 1pm

General Instructions

This is an open book exam. You may use any book and notes that you want, but no exercises, exams, or solution manuals are allowed. Solutions and answers to the problems should be well motivated. The credit for each problem is indicated in the problem. The total number of credits is 25 points. Preliminary grade limits:

Grade 3:	12 points
Grade 4:	17 points
Grade 5:	22 points

Results

The results of the exam will be posted at the latest January 25 on the course home page.

1. You are an astronaut just landed on a newly found planet and want to determine the gravitational constant g. You therefore drop an object and record the travelled distance versus time giving

$$y(t) = y_0 + gt^2/2 + e(t),$$

where e denotes noise.

a. Derive normal equations for least squares estimation of the unknowns y_0 and g if you are given N measurements of $y(t)$, $t = t_1, \dots t_N$. (2 p)

b. Describe a recursive version of the estimation algorithm, showing how the estimates of y_0 and g can be updated when a new measurement $t_{N+1}, y(t_{N+1})$ arrives. (2 p)

Solution

a. The parameters appear linearly in the equations so standard least squares estimation can be applied. This gives

$$\begin{aligned} Y &= \begin{bmatrix} y(t_1) \\ \vdots \\ y(t_N) \end{bmatrix} \\ A &= \begin{bmatrix} 1 & t_1^2/2 \\ \vdots & \vdots \\ 1 & t_N^2/2 \end{bmatrix} \\ X &= \begin{bmatrix} y_0 \\ g \end{bmatrix} \\ Y &= AX \quad \Rightarrow \quad \widehat{X} = (A^T A)^{-1} A^T Y. \end{aligned}$$

In Matlab this can be written $X = A \backslash Y$

b. See book.

2. Describe an indirect adaptive controller for the system

$$\frac{B(q)}{A(q)} = \frac{b_0}{q^2 + a_1 q + a_2}$$

that achieves model following with

$$\frac{B_m(q)}{A_m(q)} = \frac{b_{m0}}{q^2 + a_{m1} q + a_{m2}}.$$

Describe how to estimate the parameters a_1, a_2, b_0 and how to obtain the controller $R(q)u = -S(q)y + T(q)u_c$. Integral action should be used, i.e. $R(q) = (q-1)R_1(q)$ for some $R_1(q)$. Use $A_o(q) = q^n$ with minimal n. What degrees will R,S,T have? (3 p)

Solution

We can find the parameters from standard RLS using the linear relation $y(k+2) = -a_1 y(k+1) - a_2 y(k) + b_0 u(k)$.

Since we have a 2nd order system and we want integral action, we need a degree of A_o equal to $\deg(A) - 1 - \deg(B^+) + 1 = 2 - 1 - 0 + 1 = 2$. Hence $A_o = q^2$. With $R(q) = (q-1)(q+r_1)$ and $S(q) = s_0 q^2 + s_1 q + s_2$ we get the Diophantine equation

$$(q^2 + a_1 q + a_2)(q-1)(q+r_1) + b_0(s_0 q^2 + s_1 q + 2_2) = (q^2 + a_{m1} q + a_{m2})q^2$$

This gives 4 linear equations for the 4 unknowns:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ a_1 - 1 & b_0 & 0 & 0 \\ a_2 - a_1 & 0 & b_0 & 0 \\ -a_2 & 0 & 0 & b_0 \end{bmatrix} \begin{bmatrix} r_1 \\ s_0 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} a_{m1} - a_1 + 1 \\ a_{m2} + a_1 - a_2 \\ a_2 \\ 0 \end{bmatrix}$$

from which one can easily find the R and S polynomials (as long as $b_0 \neq 0$). The T polynomial is given by

$$T(q) = b_{m0} A_0(q)/b_0 = b_{m0} q^2/b_0.$$

3. Consider the system

$$\begin{aligned} x(k+1) &= x(k) + v(k) \\ y(k) &= x(k) + e(k) \end{aligned}$$

where $v \in N(0,1)$, $e \in N(0,\sigma)$ are uncorrelated with zero mean.

a. Describe the stationary Kalman filter that gives $\hat{x}(k \mid k-1)$ for the system. Is the performance better/worse/the same as that of an exponential filter

$$\hat{x}(k+1) = (1-\alpha)\hat{x}(k) + \alpha y(k)$$

with well chosen α? (2 p)

b. Determine the scalar stationary covariance P and the stationary gain $K = APC^T(\sigma_e + CPC^T)^{-1}$ as functions of σ. (2 p)

Solution

a. The stationary Kalman filter without direct term is given by

$$\hat{x}_{k+1|k} = \hat{x}_{k|k-1} + K(y_k - \hat{x}_{k|k-1})$$

It is an exponential first order filter, the performance is hence the same.

b.

$$
\begin{aligned}
P &= P+1-P^2(\sigma+P)^{-1} \Longrightarrow \quad P=\frac{1}{2}+\sqrt{\frac{1}{4}+\sigma} \\
K &= \frac{P}{\sigma+P}=\frac{1}{P}\frac{1}{\frac{1}{2}+\sqrt{\frac{1}{4}+\sigma}}
\end{aligned}
$$

P is increasing and K is decreasing as functions of σ.

4. Explain the terms prediction horizon and control horizon in MPC. Why is usually the prediction horizon chosen larger than the control horizon? Also describe which tuning parameters in Model Predictive Control (MPC) can be used to increase the speed of the closed loop system. (3 p)

Solution

See the book. If the prediction horizon is shorter than the control horizon, some of the control signals do not influence the loss function. The optimization problem then becomes undetermined and no information about these control signals are obtained.

a. One can decrease the penalty on the control signal or increase the penalty on the states. If an estimator is used to estimate unmeasured states, one can also decrease the noise level on the measurements or increase the noise level on the state noise.

5. Find the optimal 2-step ahead predictor $\hat{y}_{k+2|k}$ for the system

$$(q+0.8)y_k=(q+0.5)w_k$$

where w_k is a white noise sequence with zero mean and unit variance. Also find the variance of the resulting estimation error $E(y_{k+2}-\hat{y}_{k+2|k})^2$. (3 p)

Solution

The equation $C=AF+z^{-2}G$ gives

$$1+0.5z^{-1}=(1+0.8z^{-1})(1-0.3z^{-1})+z^{-2}0.24$$

Hence the optimal predictor is

$$\hat{y}_{k+2|k}=\frac{G}{C}=\frac{0.24}{1+0.5z^{-1}}y_k$$

and the variance equals

$$\sum f_k^2=1+(-0.3)^2=1.09.$$

6. Describe the situations where iterative learning control can be used favorably. Also describe the tradeoffs involved in choosing the filters Q and L in the ILC algorithm

$$u_k(t)=Q(q)(u_{k-1}(t)+L(q)e_{k-1}(t)$$

(2 p)

Solution

Some existing controller should assure the closed loop system T from reference to output is stable. The control task should be repeated several times and the amount of nonrepeatable noise should be small. ILC can then be used to decrease the effect of repeatable disturbances and take preventive action before the control error occurs. The filter L should ideally be an inverse of T, but quite crude approximations can be used, such as $L = kq^n$. The filter $Q(q)$ is typically a low-pass filter decreasing the ILC effect at high frequencies where the error in the approximation of the inverse might be large. For the frequencies where $Q \approx 1$ the ILC is turned on and for frequencies where $Q \approx 0$ the ILC is turned off. Without the filter Q the control error often increases again after some initial iterations of the algorithm.

7.

a. For which integers k is the static nonlinearity $y(t) = u^k(t)$ passive from u to y? (1 p)

b. For which parameters $a \geq 0, b$ is the system $G(s) = \dfrac{b}{s+a}$ passive? (1 p)

Solution

a. We have $\int y(t)u(t)dt = \int u^{k+1}(t)dt$ which is positive for all u precisely when k is odd.

b. For $a > 0$ the system is asymptotically stable and we can use the condition $\mathrm{Re}G(i\omega) \geq 0$ for all ω. Since

$$\mathrm{Re}G(i\omega) = \mathrm{Re}\frac{b}{i\omega + a} = \frac{ab}{w^2 + a^2}$$

we see that the system is passive when $a \geq 0, b \geq 0$.

8. Consider the system

$$\begin{aligned}\dot{x}_1 &= -x_1 + g_1(x_2)\\ \dot{x}_2 &= -x_2 - g_2(x_1)\end{aligned}$$

where $|g_k(u)| \leq |u|/2$, $k = 1, 2$. The system consists of two stable first order systems with nonlinear cross-coupling. Show that the system is asymptotically stable. Hint: Use the Lyapunov candiate $V = (x_1^2 + x_2^2)/2$. (2 p)

Solution

We have $V \geq 0$ and that

$$\begin{aligned}\dot{V} &= x_1\dot{x}_1 + x_2\dot{x}_2\\ &= -x_1^2 - x_2^2 + x_1 g_1(x_2) + x_2 g_2(x_1)\\ &\leq -x_1^2 - x_2^2 + |x_1 x_2|\\ &\leq -(x_1^2 + x_2^2)/2\end{aligned}$$

where we used $|x_1x_2| \leq (x_1^2 + x_2^2)/2$. We also have that $\dot{V}(x) = 0$ only for $x = 0$. The result now follows from LaSalle's theorem.

9. When estimating the parameters b, a_1, a_2 in the linear system

$$Y = \frac{b}{z^2 + a_1 z + a_2} U$$

you find that the parameters estimates do not converge to the correct values when doing experiments with U equal to a step function or a sinusoidal function. Explain the phenomenon and what can be done to improve the situation. (2 p)

Solution

The input is persistenly exiting of order 1 for a step function and of order 2 for a sinusoidal. This is not enough for estimation of three parameters, we would need an input which is persistently exciting of order at least 3. You should try a more exciting input signal, such as a sum of several sinusoidals, a pseudorandom or a random input.