LUNDS TEKNISKA HÖGSKOLA
Lunds universitet

Institutionen för
REGLERTEKNIK

FRTN15 Predictive Control

Final Exam October 25, 2012, 8am - 1pm

General Instructions

This is an open book exam. You may use any book and notes that you want, but no exercises, exams, or solution manuals are allowed. Solutions and answers to the problems should be well motivated. The credit for each problem is indicated in the problem. The total number of credits is 25 points. Preliminary grade limits:

Grade 3:	12 points
Grade 4:	17 points
Grade 5:	22 points

Results

The results of the exam will be posted at the latest November 8 on the notice board on the first floor of the M-building and on the course home page.

1. Consider the system

$$y_{k+1} = -0.1y_k - 0.8y_{k-1} + u_k + 0.5u_{k-1} + w_{k+1} + 0.3w_k,$$

where w_k is white noise with $Ew_k w_j^T = \sigma_w^2 \delta_{kj}$.

a. Calculate the 1-step-ahead predictor and give the covariance of the output error. (2 p)

b. Give the minimum variance controller minimizing the variance of the system output. (1 p)

c. Can you still use the minimum variance controller if the system to be controlled is non-minimum phase? Motivate your answer. (1 p)

Solution

a. The Diophantine Equation for the 1-step-ahead predictor is:

$$C(z^{-1}) = A(z^{-1})F(z^{-1}) + z^{-1}G(z^{-1})$$
$$(1 + 0.3z^{-1}) = (1 + 0.1z^{-1} + 0.8z^{-2})f_0 + z^{-1}(g_0 + g_1 z^{-1})$$

With this we have

$$f_0 = 1$$
$$g_0 = 0.2$$
$$g_1 = -0.8$$

So that

$$F(z^{-1}) = 1$$
$$G(z^{-1}) = 0.2 - 0.8z^{-1}$$

And the 1-step-ahead predictor is

$$\begin{aligned}\hat{y}_{k+1|k} &= \frac{G(z^{-1})}{C(z^{-1})} y_k + \frac{B(z^{-1})F(z^{-1})}{C(z^{-1})} u_k \\ &= \frac{0.2 - 0.8z^{-1}}{1 + 0.3z^{-1}} y_k + \frac{(1 + 0.5z^{-1})}{1 + 0.3z^{-1}}\end{aligned}$$

b. The minimum variance controller is

$$\begin{aligned}u_k &= -\frac{G(z^{-1}}{B(z^{-1})F(z^{-1})} \\ &= -\frac{0.2 - 0.8z^{-1}}{1 + 0.5z^{-1}}\end{aligned}$$

c. This minimum variance controller can only be used if the B polynom is stable, i.e. it has all its roots inside the unit disk. For a non-minimum phasse system, there are zeros ouside the unit disc which become poles of the controller.

2. Consider the relation given by

$$y(t) = \arctan[ax_1(t)^2 + bx_2(t) + e(t)],$$

where e denotes noise. Derive normal equations for least squares estimation of the parameters a, b if we are given N measurements of $y(t)$, $x_1(t)$ and $x_2(t)$, $t = 1, \ldots N$. (2 p)

Solution

For the standard least squares estimate we need the parameters to appear linearly in the equations, but we can rewrite the equation as

$$\tan(y) = ax_1^2 + bx_2 + e.$$

We can now write the standard equations for our estimate as:

$$\begin{aligned} Y &= \begin{pmatrix} \tan(y(1)) & \ldots & \tan(y(N)) \end{pmatrix}^T \\ A &= \begin{pmatrix} x_1^2(1) & \ldots & x_1^2(N) \\ x_2(1) & \ldots & x_2(N) \end{pmatrix}^T \\ X &= \begin{pmatrix} a & b \end{pmatrix}^T \\ Y = AX \quad &\Rightarrow \quad \hat{X} = (A^T A)^{-1} A^T Y \end{aligned}$$

3. Describe an indirect adaptive controller for the system

$$\frac{B(q)}{A(q)} = \frac{b_0 q + b_1}{q^2 + a_1 q + a_2}$$

that achieves model following with

$$\frac{B_m(q)}{A_m(q)} = \frac{b_{m0}}{q^2 + a_{m1} q + a_{m2}}.$$

Describe how to estimate the parameters a_1, a_2, b_0, b_1 and how to obtain the controller $R(q)u = -S(q)y + T(q)u_c$. No integral action is required and the zero of the system is cancelled, i.e. $B^+ = q + b_1/b_0$ (you can assume $b_0 \neq 0$.) Use $A_o(q) = q^n$ with minimal n. What degrees will R,S,T have? (3 p)

Solution

We can find the parameters from standard RLS using the linear relation $y(k+2) = -a_1 y(k+1) - a_2 y(k) = b_0 u(k+1) + b_1 u(k)$. Since this is a 2nd order system and we cancel the zero, we need degree of A_o equal to $\deg(A) - 1 - \deg(B^+) = 2 - 1 - 1 = 0$. Hence $A_o = 1$. With $R(q) = q + r_1$ and $S(q) = s_0 q + s_1$ we get the Diophantine equation

$$(q^2 + a_1 q + a_2)(q + r_1) + (b_0 q + b_1)(s_0 q + s_1) = (q^2 + a_{m1} q + a_{m2})(q + b_1/b_0)$$

This gives 3 linear equations for the 3 unknowns. The solution is $r_1 = b_1/b_0$, $s_0 = (a_{m1} - a_1)/b_0$, $s_1 = (a_{m2} - a_2)/b_0$.

4. Consider the system

$$
\begin{aligned}
x_{k+1} &= 2x_k + u_k + v_k \\
y_k &= x_k + e_k \\
v_k &\sim \mathrm{N}(0,1) \\
e_k &\sim \mathrm{N}(0,10) \\
x_0 &\sim \mathrm{N}(0,P_0)
\end{aligned}
$$

Here $N(0,P)$ stands for a Gaussian variable with mean 0 and variance P.

a. Find a deadbeat observer for the system, i.e. one with $A-KC$ having eigenvalue in zero. (1 p)

b. Describe the optimal time-varying Kalman filter for $\hat{x}_{k|k-1}$. (1 p)

c. Assuming stationarity, what is the Kalman filter gain, K? (1 p)

d. Which of the two described observers will have the smallest estimation variance $E(\tilde{x}^2)$? (1 p)

Solution

a. With $K=2$ we get $A-KC=0$. A deadbeat observer is hence

$$\hat{x}_{k+1} = 2\hat{x}_k + u_k + 2(y_k - \hat{x}_k) = u_k + 2y_k$$

b. The time-varying Kalman filter without direct term is given by

$$
\begin{aligned}
\hat{x}_{k+1|k} &= 2\hat{x}_{k|k-1} + u_k + K_k(y_k - \hat{y}_{k|k-1}) \\
\hat{y}_{k+1|k} &= \hat{x}_{k+1|k} \\
K_k &= \frac{2P_{k|k-1}}{10 + P_{k|k-1}} \\
P_{k+1|k} &= 4P_{k|k-1} + 1 - 4\frac{P_{k|k-1}^2}{10 + P_{k|k-1}}
\end{aligned}
$$

c. We need to find the stationary value where $P_{k+1|k} = P_{k|k-1}$. This gives the solution $P \approx 31.32$ and $K \approx 1.52$

d. For the deadbeat observer without direct term we get $\tilde{x} = 2e_k + v_k$ which has variance $E(\tilde{x}^2) = 2^2 \cdot 10 + 1 = 41$. The Kalman filter without direct term gives estimation variance $E(\tilde{x}^2) = P \approx 31.32$, a clear improvement.

(Remark: Described above are observers without direct term. A deadbeat observer with direct term is given by

$$\hat{x}_{k+1} = y_{k+1}$$

it gives eigenvalues of $A - AKC$ in the origin. This deadbeat observer with direct term has $E(\tilde{x}^2) = E(e^2) = 10$. The Kalman filter with direct term has variance $P_{k|k} = (A-KC)P_{k|k-1} = 7.6$.)

5. Describe how one can handle the situation when the optimization in the MPC controller calculation shows that no control sequence exists that satisfies all the constraints. (1 p)

Solution

One can relax the constraints, making them soft. One can use different weights on the different constraints finding a controller that makes a weighted combination of the constraints violations as small as possible. See Section 13.2.4 in the book.

6.

a. Is the relay system described by $y = \text{sign}(u)$ passive from u to y? (2 p)

b. For which parameters a is the system $G(s) = \dfrac{s+a}{(s+1)^2}$ passive? (2 p)

Solution

a. Yes. We have $\int_0^T y(t)u(t)dt = \in \beta^T |u(t)|dt \geq 0$.

b. The condition for PR is that $\operatorname{Re} G(i\omega) \geq 0$ for all ω. We have

$$\operatorname{Re} G(i\omega) = \operatorname{Re} \frac{i\omega + a}{(i\omega + 1)^2} = \operatorname{Re} \frac{(i\omega + a)(-i\omega + 1)^2}{(\omega^2 + 1)^2} = \frac{\omega^2(2-a) + a}{(\omega^2+1)^2}$$

which shows that G is PR precisely when $0 \leq a \leq 2$.

7. An integrator system

$$\dot{y}(t) = ku(t)$$

with unknown gain $k > 0$ should be controlled with a zero-order continuous-time controller

$$u(t) = -\alpha(t)y(t) + \beta(t)u_c(t)$$

The desired response is given by

$$\dot{y}_m(t) = -a_m y_m(t) + b_m u_c(t)$$

Use Lyapunov theory to find a parameter update law ($\dot{\alpha} = \ldots,\ \dot{\beta} = \ldots$) of an adaptive controller guaranteeing that the error $e = y - y_m$ goes to zero. Try the Lyapunov candidate

$$V(x) = \frac{1}{2}\left(e(t)^2 + \frac{1}{k}(k\alpha(t) - a_m)^2 + \frac{1}{k}(k\beta(t) - b_m)^2\right)$$

(3 p)

Solution

We have $V \geq 0$ and

$$\begin{aligned}
\dot{V} &= e\dot{e} + (k\alpha - a_m)\dot{\alpha} + (k\beta - b_m)\dot{\beta} \\
&= e(ku + a_m y_m - b_m u_c) + (k\alpha - a_m)\dot{\alpha} + (k\beta - b_m)\dot{\beta} \\
&= e(-k\alpha y + k\beta u_c + a_m y_m - b_m u_c) + (k\alpha - a_m)\dot{\alpha} + (k\beta - b_m)\dot{\beta} \\
&= -a_m e^2 + (k\alpha - a_m)(\dot{\alpha} - ey) + (k\beta - b_m)(\dot{\beta} + eu_c)
\end{aligned}$$

We see that by choosing the parameter update law $\dot{\alpha} = ey$ and $\dot{\beta} = -eu_c$ we get

$$\dot{V} = -a_m e^2 \leq 0$$

This proves that e goes to zero.

8. We want the output y of the discrete-time system $Y(z) = G(z)U(z)$ with

$$G(z) = \frac{z+2}{z(z+0.5)}$$

to follow a given reference signal y_r. In this exercise we assume y_r is a step function, ie $y_r(t) = 1, \ t \geq 0$

a. The following ILC algorithm (with $Q = 1$)

$$u_{k+1}(t) = u_k(t) + \gamma q^n e_k(t)$$

can be used to find an input signal $u(t) = \lim_{k\to\infty} u_k(t)$ that solves the problem. Figures 1a-d illustrate the amplitude curves $|I - \gamma z^n G(z)|_{z=e^{j\omega}}$ for some different values of γ and n. Also shown is the successful result of 100 iterations of the ILC algorithm for one of these choices of parameters, the other three parameter combinations do not work. Which parameter set is the working one (a,b,c or d)? (1 p)

b. Calculating U as

$$U(z) = G^{-1}(z)Y_r(z) = \frac{z(z+0.5)}{z+2}Y_r(z)$$

and using the inverse Z-transform to obtain u does not work. Why? (1 p)

c. Describe how to calculate a bounded control signal, $u(t), \ -\infty < t < \infty$, that will make the output equal to a step function (without using ILC iteration). *Hint. Split the inverse of $G(z)$ into causal and anti-causal parts. Describe how to interpret the different parts as stable recursions to find u.* (2 p)

Solution

a. The figure in c) is the only having amplitude less than 1. It is the correct one.

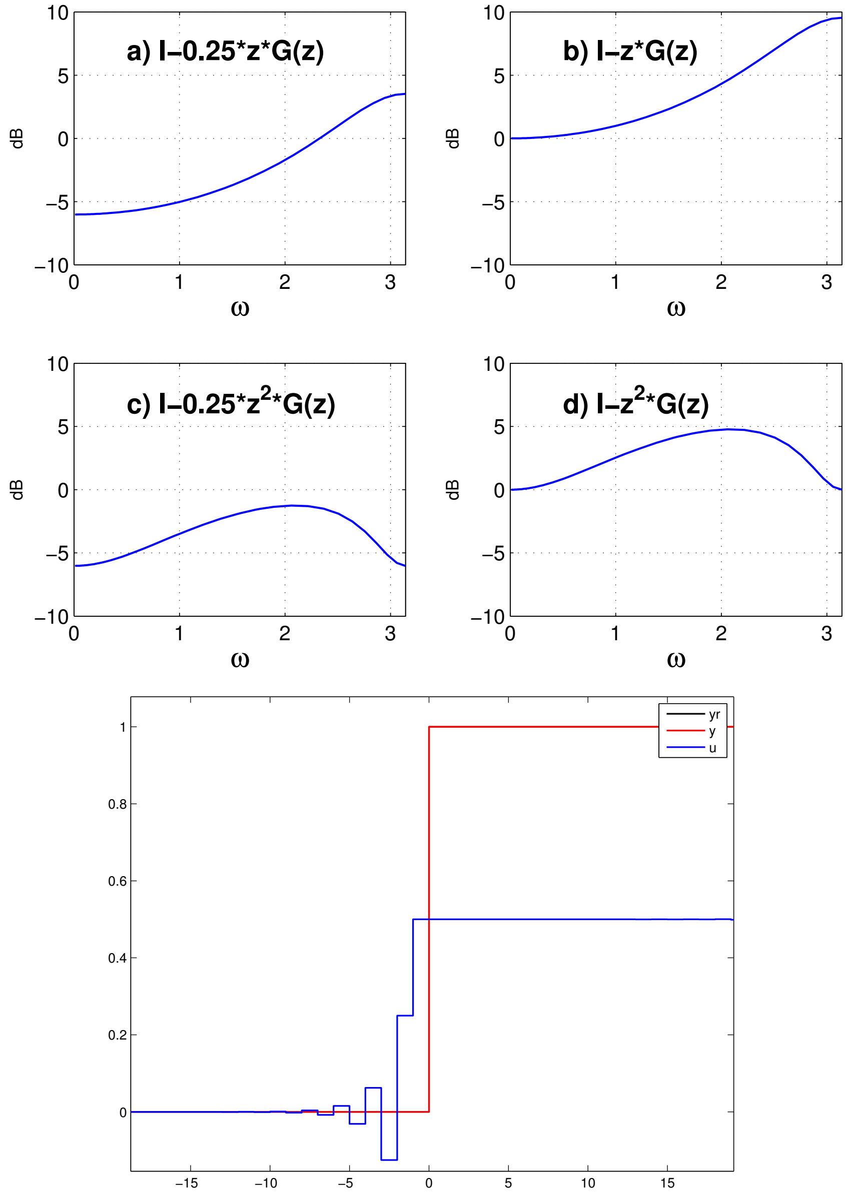
b. The system can not simply be inverted because of the zero outside the unit circle. The calculated control signal will contain an unbounded term of the form $(-2)^k$.

c. An expansion of the inverse gives:

$$G^{-1}(z) = \frac{z(z+0.5)}{z+2} = z - 1.5 + \frac{3}{z+2}$$

This means that $u(t) = y_r(t+1) - 1.5y_r(t) + x(t)$ where

$$x(t+1) + 2x(t) = 3y_r(t).$$



Figur 1 The ILC amplitude curves a-d (top) for different combinations of γ and n. The successful result after 100 iterations of the ILC algorithm (bottom).

This recursion can be rewritten as a stable anticausal filter

$$x(t) = -0.5x(t+1) + 1.5y_r(t).$$

The solutions to this with $y_r(t) = 1$ for $t \geq 0$ is $x(t) = (-2)^t$, when $t \leq 0$, and $x(t) = 1$ for $t \geq 0$. The solution is hence

$$u(t) = (-2)^t, t \leq -2, \quad u(t) = 0.5, t \geq -1.$$

which is very close to the control signal generated by the ILC and shown in the figure.