FRTN15 Predictive Control—Exercise 6

1. Show that the tracking error fulfills the recursive equation
$e_k(t) = [(1 - Q(q))(1 - T_c(q))]y_d(t) + [Q(q)(1 - L(q)T_c(q))]e_{k-1}(t)$
on lecture 11. What happens if $Q = 1$? If $Q \neq 1$?.

Solution

$$
\begin{aligned}
y_k(t) &= T_c(q)y_d(t) + T_c(q)u_k(t) \\
e_k(t) &= y_d(t) - y_k(t) \\
u_k(t) &= Q(q)[u_{k-1}(t) + L(q)e_{k-1}(t)] \\
\implies e_k(t) &= (1 - T_c(q))y_d(t) - T_c(q)u_k(t) \\
&= (1 - Tc(q))y_d(t) - T_c(q)Q(q)[u_{k-1}(t) + L(q)e_{k-1}(t)] \\
&= (1 - T_c(q))y_d(t) - Q(q()T_c(q)u_{k-1}(t) - Q(q)T_c(q)L(q)e_{k-1}(t) \\
&= (1 - Tc(q))y_d(t) - Q(q()T_c(q)u_{k-1}(t) \\
&\quad -Q(q)T_c(q)L(q)e_{k-1}(t) - Q(q)y_d(t) + Q(q)y_d(t) \\
&= (1 - Q(q))(1 - Tc(q))y_d + Q(q)y_d - Q(q)y_{k-1} - Q(q)T_c(q)L(q)e_{k-1} \\
&= (1 - Q(q))(1 - Tc(q))y_d + Q(q)(1 - Tc(q)L(q))e_{k-1}
\end{aligned}
$$

For $Q = 1$ there will be no residual error, for any $Q \neq 1$ there will be a residual error assuming that $T_c(q)\ ne1$. If $T_c(q) = 1$ ILC would be pointless since the closed loop system would be easily inverted and a perfect u could be found without iterating.

2. Consider the system

$$G(q) = \frac{0.09516}{q - 0.9048}.$$

It is controlled using ILC (see Figure 1) such that the control signal at an iteration k is given by:

$$u_{k+1}(t) = u_k(t) + L(q)e_k(t)$$

where $e_k(t) = r(t) - y_k(t)$.
Study the convergence of the ILC iterations for $L(q) = 1$ and $L(q) = q$.
Hint: The Nyquist plots of $G(q)L(q)$ for the two chosen L are shown in Figure 2.

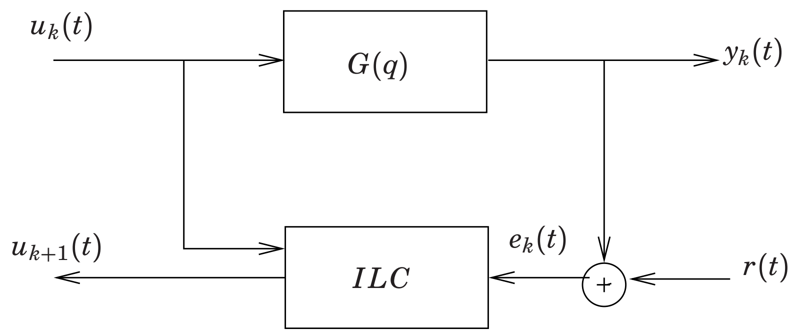
Solution
The error between iteration is obtained as:

$$
\begin{aligned}
e_{k+1}(t) = r(t) - y_{k+1}(t) = r(t) - G(q)u_{k+1}(t) \\
=r(t) - G(q)u_k(t) - G(q)L(q)e_k(t) = (1 - G(q)L(q))e_k(t)
\end{aligned}
$$

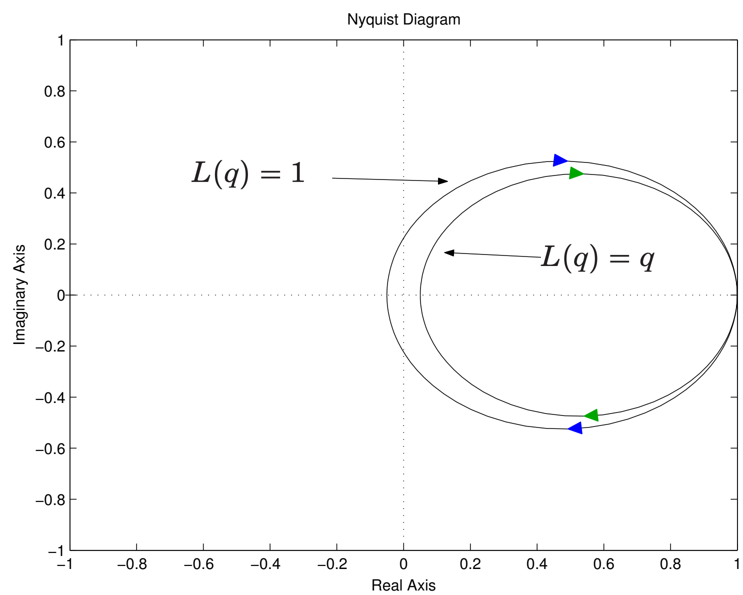
The condition for the error not to grow is:

$$|1 - G(e^{i\omega})L(e^{i\omega})| < 1, \quad \forall \omega \in [-\pi,\ \pi]$$

From the Nyquist-plot results that this is not the case for $L(q) = 1$.



Figur 1 AN ILC feedback system.



Figur 2 Nyquist plots for $G(q)L(q)$.

3.

a. Show that the system

$$\begin{aligned} \dot{x} &= -x + u, \quad x(0) = x_0, && (1) \\ y &= x && (2) \end{aligned}$$

with transfer function

$$G_1(s) = \frac{1}{(s+1)} \tag{3}$$

is strictly positive real (SPR) and that the storage function

$$V(x) = \frac{1}{2}x^T x$$

fulfills the passivity property

$$V(x(t)) = V(x(0)) + \int_0^t y^T(\tau)u(\tau)dt - \int_0^t x^T(\tau)x(\tau)d\tau \tag{4}$$

What is the interpretation of all the three terms on the right-hand side of Eq. (4)?

b. Show that the transfer function

$$G_2(s) = \frac{1}{(s+1)^2} \tag{5}$$

is not positive real.

Solution

a. A transfer function $G(s)$ is said to be *Positive Real* (PR) if:

$$\text{Re } G(s) \geq 0 \text{ for Re } s \geq 0$$

The transfer function is said to be *Strictly Positive Real* (SPR) if $G(s - \epsilon)$ is positive real for some $\epsilon > 0$. In terms of the Nyquist diagram, an SPR system must have a Nyquist curve which lies strictly in the right half-plane. For the case with

$$G(s) = \frac{1}{(s+1)}$$

we note that Re $G(s - \epsilon) \geq 0$ for Re $s \geq 0$ for any $\epsilon \leq 1$, which proves that the system is SPR.

To show that the storage function (Lyapunov function) given by:

$$V(x) = \frac{1}{2}x^T x$$

fullfills the passivity property, we begin by finding the time derivative:

$$\begin{aligned}
\dot{V}(x(t)) &= \frac{1}{2}(\dot{x}^T x + x^T \dot{x}) \\
&= \frac{1}{2}((-x+u)^T x + x^T(-x+u)) \\
&= -x^T x + x^T u
\end{aligned}$$

Since $y(t) = x(t)$ we may write:

$$\dot{V}(x(t)) = -x^T x + y^T u$$

At a specific time T the storage function is given by:

$$\begin{aligned}
V(x(T)) &= V(x(0)) + \int_0^T \dot{V}(x(t))dt \\
&= V(x(0)) + \int_0^T (-x^T x + y^T u)dt
\end{aligned}$$

Thus we obtain the passivity property:

$$V(x(T)) = V(x(0)) + \int_0^T y^T u dt - \int_0^T x^T x dt$$

as required. The term $V(x(0))$ represents the initial stored energy at time $t = 0$. The term $\int_0^T y^T u dt$ represents the energy supplied to the system between time $t = 0$ and $t = T$. To see this more clearly, it may be helpful to think of an electrical system where the input u is a voltage and the output y is a current. Power is given by $P = VI$ and the energy supplied to the system is thus the integral of this. The last term $\int_0^T x^T x$ represents the energy dissipated by the system. Returning to the electrical system example, we know that dissipated power is given by I^2/R, so the dissipated energy is therefore the integral of this.

In summary, a system is passive if the change in stored energy is equal to the energy supplied minus the energy dissipated.

b. By looking at the Nyquist plot of the system we see that Re $G(s)$ takes negative values for certain ω when Re $s = 0$. The system is therefore not positive real.

4. Ex 12.2, Predictive and Adaptive Control

Consider Iterative Learning Control (ILC) given by the equations:

$$\begin{aligned} y_k(t) &= G_v(q)u_k(t) \\ e_k(t) &= r(t) - y_k(t) \\ u_k(t) &= Q(q)[u_{k-1}(t) + L(q)e_{k-1}(t)] \end{aligned}$$

where $G_c(q)$ is the closed-loop transfer function of the system and q is the forward time shift operator. Assume that $Q(q) = 1$ and that

$$G_c(q) = \frac{1}{(q-0.7)(q-0.9)}, L(q) = k(q-0.5)(q-0.7)(q-0.9)$$

where k is a positive constant. Determine a condition on k which, if fulfilled, guarantees that the error of the resulting ILC scheme converges.

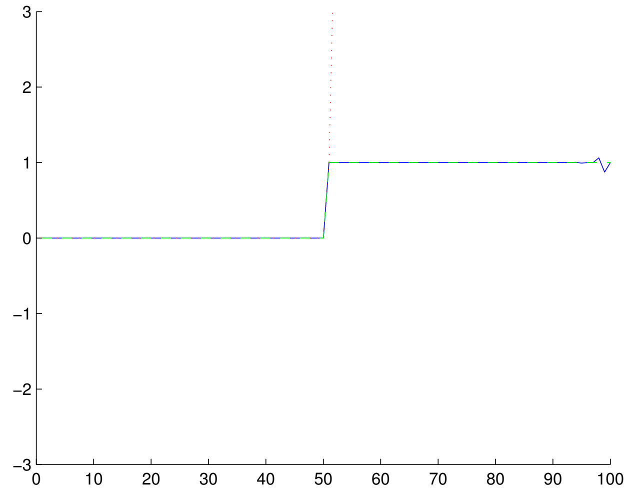
Solution

$$Q(q) = 1 \implies e_k(t) = r(t) - G_c(q)[u_{k-1}(t) + L(q)e_{k-1}(t)] = [1 - G_c(q)L(q)]e_{k-1}(t)$$

Stable if $\sup\limits_{\omega h \in [-\pi,\pi]} ||1 - G_c(q)L(q)||_{z=e^{i\omega h}} < 1$

$$\begin{aligned} \sup_{\omega h \in [-\pi,\pi]} ||1 - G_c(q)L(q)||_{z=e^{i\omega h}} &= \sup_{\omega h \in [-\pi,\pi]} ||1 - k(e^{i\omega h} - 0.5)|| = \\ &\sup_{\omega h \in [-\pi,\pi]} ||1 - k(\cos wh + i \sin wh - 0.5)|| = \\ &\sup_{\omega h \in [-\pi,\pi]} ||1 - k(\cos wh + i \sin wh - 0.5)|| = \\ &\sup_{\omega h \in [-\pi,\pi]} \sqrt{(0.5 - k\cos wh)^2 + k^2 \sin wh} = \\ \sup_{\omega h \in [-\pi,\pi]} \sqrt{0.5^2 - 2k\cos wh + k^2} &= \sqrt{0.5^2 - 2k + k^2} < 1 \Leftarrow \end{aligned}$$

$$(k+1)^2 - \frac{3}{4} < 1 \Leftarrow k + 1 < \sqrt{\frac{7}{4}} \Leftrightarrow k < \frac{\sqrt{7}}{2} - 1$$



Figur 3 Unbounded u using a causaul filter.

5. Dead-beat ILC

Consider the system

$$y = G(q)u = \frac{q-2}{(q+0.5)(q+0.9)}u$$

Describe how to interpret the formula

$$u = \frac{(q+0.5)(q+0.9)}{q-2}y_r$$

as a non-causal filter giving a bounded signal u fullfilling $y = y_r$, where y_r is a given reference value. Simulate with y_r equal to e.g. a step function.

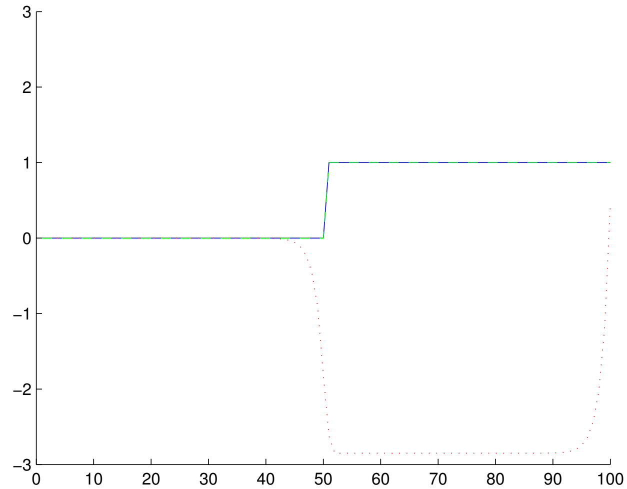
Solution

Inverting $G(q)$ gives $G^{-1}(q) = \dfrac{(q+0.5)(q+0.9)}{q-2}$ which is an unstable transfer function, using this to calculate u leads to an unbounded signal. We can however rewrite the transfer to instead work backwards in time, ie. an anti-causal filter.

$$y(k+2) + 1.4y(k+1) + 0.45y(k) = u(k+1) - 2u(k) \implies$$
$$u(k) = 0.5u(k+1) - 0.5[y(k+2) + 1.4y(k+1) + 0.45y(k)]$$

Running this system backwards in time it is stable!
Introduce $\hat{u}(k) = u(t_f + 1 - k), y(\hat{k}) = y(t_f + 1 - k)$

$$\begin{aligned} \hat{u}(tf+1-k) &= 0.5\hat{u}(tf-k) - 0.5[\hat{y}(tf-1-k) \\ &\quad +1.4\hat{y}(tf-k) + 0.45\hat{y}(tf+1-k)] \\ \implies \hat{u}(k+2) &= 0.5\hat{u}(k+1) - 0.5[\hat{y}(k) \\ &\quad +1.4\hat{y}(k+1) + 0.45\hat{y}(k+2)] \\ \implies q(q-0.5)\hat{u} &= -0.5(1 + 1.4q + 0.45q^2)\hat{y} \end{aligned}$$



Figur 4 Bounded u using an anti-causaul filter.

which is a stable filter! (all poles inside the unit circle)

See Fig. 4 and Fig. 3 for the anti-causal and the causal filter, notice both control signals follow the reference value but the causal filter leads to an unbounded control signal.

Notice that if we had had zeros both inside and outside the unit circle we could have done a partial fractional expansion of the system collecting all the 'stable' zeros in one term, all the 'unstable' zeros in another and in the general case also a direct term. We would then filter the stable inverses in forward time and the unstable inverse in backwards time.

Matlab Code

```
yr = [zeros(1,50), ones(1,50)]
q = tf('q',1)
Gq = (q-2)/((q+0.5)*(q+0.9))

% Zero outside the unit circel imples that the inverse
    of Gq would be
% unstable. We can however rewrite it as a stable
   anticausal filter!
% y(k+2) + 1.4 y(k+1) + 0.9*0.5*y(k) = u(k+1) - 2u(k)
% => u(k) = 0.5*u(k+1) - 0.5*y(k+2) - 0.5*1.4 y(k+1) -
    0.5*0.5*0.9*y(k)
% Backwards in time this is a stable system!
% k1 = k => k2 = tf+1-k => k2=tf+1-k1 => k1=tf+1-k2
% ub(k) = u(tf+1-k) => u(k) = ub(tf+1-k)
% yb(k) = y(tf+1-k) => y(k) = yb(tf+1-k)
% => ub(tf+1-k) = 0.5*ub(tf-k) - 0.5yb(tf-1-k) -
   0.5*1.4yb(tf-k) - 0.5*0.5*0.9*yb(tf+1-k)
```

```
% => ub(k+2) = 0.5*ub(k+1) - 0.5yb(k) - 0.5*1.4yb(k+1)
    - 0.5*0.5*0.9*yb(k+2)
% => (q^2-0.5q)*ub = -0.5*(1 + 1.4q + 0.5*0.9q^2)*yb
% ub(k) = ub2(k-1) =>
% (q-0.5)*ub2 = -0.5*(1 + 1.4q + 0.5*0.9q^2)*yb

a = [-0.5 1];
b = -0.5*[1 1.4 0.5*0.9];

arev = [1 -0.5];
brev = -0.5*[ 0.9*0.5 1.4 1];

ub2=filter(brev,arev,yr(end:-1:1),[1 1]);
u=[0 ub2(end:-1:2)];
a = Gq.den{1}
b = Gq.num{1}
b = b(2:end)

y = filter(b,a,u,[0 0])

figure(1)
clf
hold on
plot(y,'b')
plot(u, 'r:')
plot(yr,'g--')
axis([0 100 -3 3])
figure(2)
u = filter(a,b,yr,[0 0]);
y = filter(b,a,u,[0 0]);
clf
hold on
plot(y,'b')
plot(u, 'r:')
plot(yr,'g--')
axis([0 100 -3 3])
```