LUNDS TEKNISKA HÖGSKOLA
Lunds universitet

Institutionen för
REGLERTEKNIK

FRTN15 Predictive Control

Final Exam October 17, 2011, 8am - 1pm

General Instructions

This is an open book exam. You may use any book you want, but no notes, exercises, exams, or solution manuals are allowed. Solutions and answers to the problems should be well motivated. The exam consists of 7 problems. The credit for each problem is indicated in the problem. The total number of credits is 25 points. Preliminary grade limits:

Grade 3:	12 – 16 points
Grade 4:	17 – 21 points
Grade 5:	22 – 25 points

Results

The results of the exam will be posted at the latest October 24 on the notice board on the first floor of the M-building.

1.

a. Consider the nonlinear model:

$$y(t) + a_1 y(t-1) = b_1 u(t-1) + b_2 u(t-1)y(t-1)$$

Find a linear-in-parameters regression model for estimation of the parameters a_1, b_1 and b_2. (1 p)

b. It is desired to use this regression model to perform online identification of the parameters. In addition, it is known that the parameters may be slowly time-varying. Write down an appropriate algorithm for this estimation task, and explain its operation. (1 p)

c. An indirect adaptive controller is to be designed, using the estimation method from above. Design a control law, incorporating the reference signal $u_c(t)$, such that the closed-loop system has the transfer function:

$$Y(z) = \frac{b_0}{z + a_0} U_c(z)$$

where $1 > a_0 > -1$. The controller may be nonlinear. (1 p)

Solution

a. The output is given by:

$$y(t) = -a_1 y(t-1) + b_1 u(t-1) + b_2 u(t-1)y(t-1)$$

The regression model must be linear in the parameters, a condition satisfied by choosing:

$$\theta = [\, a_1 \quad b_1 \quad b_2 \,]^T$$
$$\phi^T(t) = [\, -y(t) \quad u(t) \quad u(t)y(t) \,]$$

The output is thus given by:

$$y(t) = \phi^T(t-1)\theta$$

b. The estimation of time-varying parameters can be implemented as a Kalman filter:

$$\hat{\theta}_k = \hat{\theta}_{k-1} + K_k \epsilon_k$$
$$K_k = \frac{P_{k-1}\phi_k}{R_2 + \phi_k^T P_{k-1}\phi_k}$$
$$\epsilon_k = y_k - \phi_k^T \hat{\theta}_{k-1}$$
$$P_k = P_{k-1} - \frac{P_{k-1}\phi_k\phi_k^T P_{k-1}}{R_2 + \phi_k^T P_{k-1}\phi_k} + R_1$$

As an alternative, we may choose the Recursive Least Squares algorithm, with exponential forgetting factor.

c. The output is given by:

$$y(t) = -a_1 y(t-1) + b_1 u(t-1) + b_2 u(t-1) y(t-1)$$

The desired response is given by:

$$y(t) = -a_0 y(t-1) + b_0 u_c(t-1)$$

Equating these gives:

$$-a_0 y(t-1) + b_0 u_c(t-1) = -a_1 y(t-1) + b_1 u(t-1) + b_2 u(t-1) y(t-1)$$
$$u(t) = \frac{(a_1 - a_0) y(t) + b_0 u_c(t)}{b_1 + b_2 y(t)}$$

2. A self-tuning regulator using an RLS estimation algorithm has been designed for a second order system with unknown parameters. During testing, simulations have been carried out for different values of the forgetting factor λ and the measurement noise variance σ^2. Figure 1 shows the parameter estimates during a process variation test, where the unknown system's dynamics are changes at $t = 50$. The test was carried out for four different conditions:

1. $\lambda = 0.8, \ \sigma^2 = 0.001$

2. $\lambda = 0.9, \ \sigma^2 = 0.001$

3. $\lambda = 0.9, \ \sigma^2 = 0.0001$

4. $\lambda = 0.95, \ \sigma^2 = 0.001$

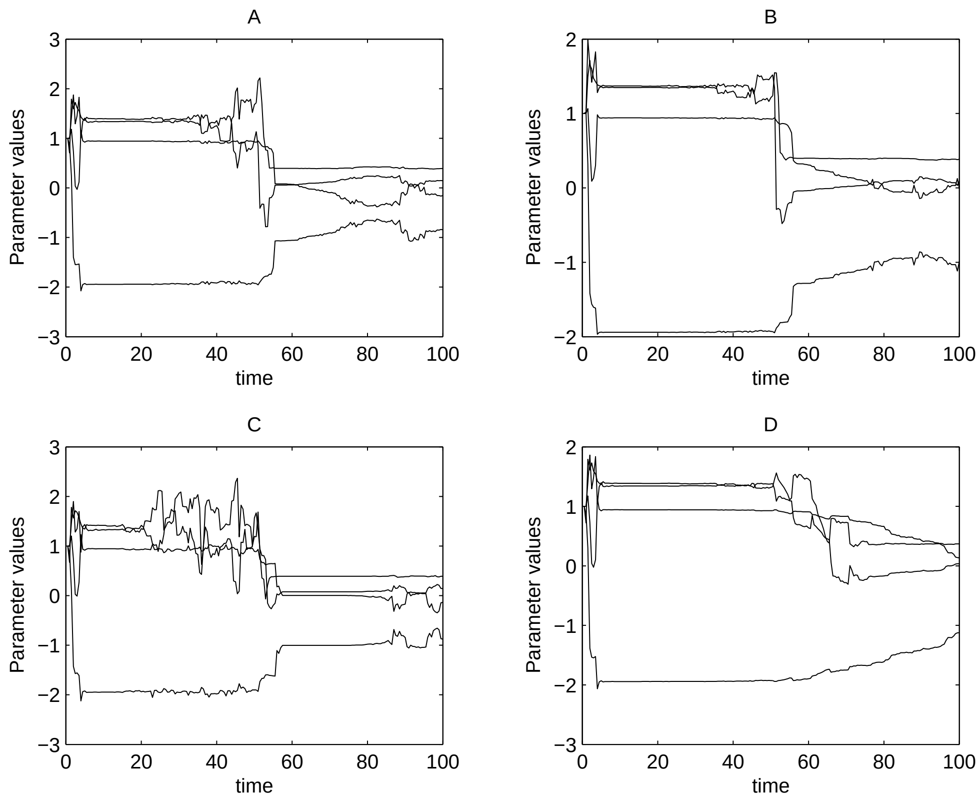
Unfortunately, the engineer responsible for the tests was not very methodical and forgot to write down the conditions corresponding to each of the results.

a. Assist the engineer by determining which of the cases 1–4 above correspond to the plots A–D in Figure 1. Clearly state your reasoning. (2 p)

b. The engineer is not pleased with the noise performance of the system with lower values of the forgetting factor. Is it possible to achieve better noise rejection by adjusting the value of the initial covariance matrix P_0? Explain your answer. (1 p)

Solution

a. The problem can be solved by considering two of the properties of the forgetting factor. Firstly, a larger forgetting factor gives slower convergence after parameter changes. Secondly, a large forgetting factor is useful for reducing the effects of noise. To begin with, we see that the largest forgetting factor is 0.95, in case 4. The slowest parameter convergence occurs in plot D, so we conclude that case 4 corresponds to plot D. The smallest forgetting



Figur 1 Parameter plots for Problem 2

factor occurs in case 1, where a high noise influence is present. Therefore it is reasonable to associate this case with the plot which exhibits the most noise, namely plot C. This leaves two cases, 2 and 3, which both involve the same forgetting factor value but different noise variance values. Examining the noise presence in the remaining plots A and B leads us to the conclusion that case 2 corresponds to plot A and case 3 correspnds to plot B. The answer is therefore:
A–2
B–3
C–1
D–4

b. No, it is not possible. The initial covariance matrix only has an effect on the initial behaviour of the system; thereafter it plays no part in noise rejection. In order to reduce the effect of noise, regression filters could be introduced.

3. Consider the process

$$y_k = \frac{1}{z - a_u} u_k + \frac{z}{z - a_e} e_k$$

where the noise sequence $\{e_k\}$ is independent white noise and u_k is the control signal.

a. Design a controller that minimizes $E(y_k^2)$. (2 p)

b. Assume that the parameter a_u is known and that a_e is unknown. Derive a regression model for identification of the parameter a_e. (1 p)

c. Design a controller for the process when a_u is known and a_e is slowly time-varying with values in a large interval. Motivate your choice of controller structure. (2 p)

Solution

a. The model can be written as $A(z)y_k = B(z)u_k + C(z)e_k$ where

$$A(z) = (z - a_u)(z - a_e), \quad B(z) = z - a_e, \quad C(z) = z(z - a_u).$$

Although A and C have common factors, we have:

$$\frac{C(z)}{A(z)} = 1 + \frac{(z - a_u)a_e}{A(z)} = F(z) + \frac{G(z)}{A(z)}$$

the minimum variance control law is

$$u_k = -\frac{G(z)}{B(z)F(z)}y_k = -\frac{(z - a_u)a_e}{z - a_e}y_k.$$

b. Introducing

$$\bar{y}_k = y_k - \frac{1}{z - a_u}u_k$$

we find that

$$\bar{y}_{k+1} = \bar{y}_k a_e + e_{k+1}.$$

A regression model is

$$\bar{y}_{k+1} = \bar{y}_k a_e.$$

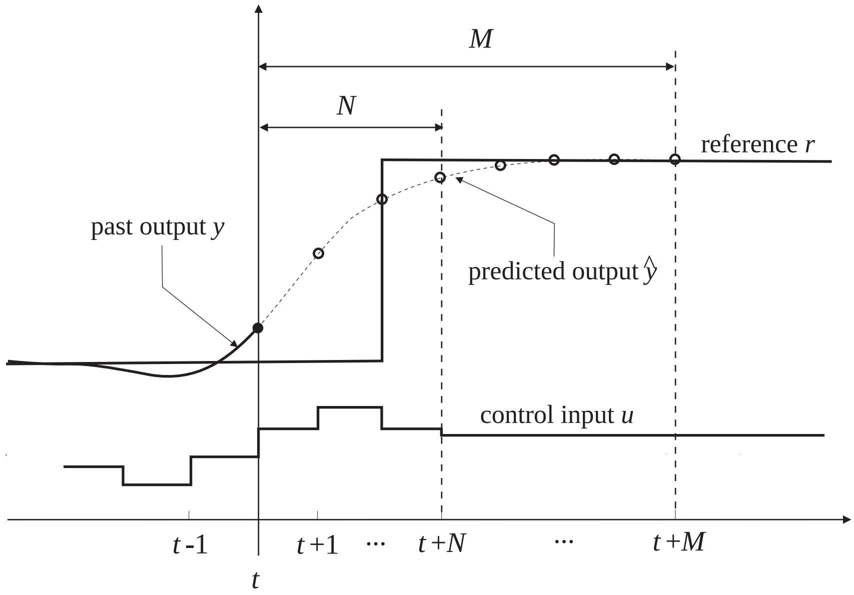
c. Since a_e is time-varying with values in a large interval it seems reasonable to use an adaptive controller, *e.g.* an indirect self-tuning regulator. The identification algorithm must have some kind of forgetting or resetting since a_e is time-varying. The obtained estimates are used to calculate the controller parameters.

4.

a. Briefly explain the principle of Model Predictive Control. Use a sketch or a diagram to illustrate the terms *prediction horizon* and *control horizon*. (1 p)

b. What computational problems can arise if the plant is operated near constraints on the outputs, and how can the MPC formulation be modified to limit these problems? (1 p)

Solution



Figur 2 Illustration of the MPC principle for Problem 4.

a. The receding horizon principle is illustrated in Figure 2. Given the current state measurement (or estimate), a sequence of future control inputs are determined by minimizing a cost function penalizing predicted inputs and outputs of the system. The first input is implemented, and the optimization is repeated at the next sample.

b. Problems may arise if an output constraint is violated, or the current state of the system doesn't allow a feasible solution to be found. This may be caused by e.g. disturbances or estimation errors when state measurements are not available.

One way of limiting these issues is to use 'soft' constraints on outputs and states. For instance, a constraint on the form $x_{\text{min}} \leq x(k) \leq x_{\text{max}}$ can be replaced by $x_{\text{min}} - \epsilon(V_k^x)_{\text{min}} \leq x(k) \leq x_{\text{max}} + \epsilon(V_k^x)_{\text{max}}$ where ϵ is a slack variable and $(V_k^x)_{\text{min}}$ and $(V_k^x)_{\text{max}}$ are relaxation vectors. An extra term may then be added to the cost function that penalizes ϵ^2. This allows the constraints to be violated but at a high cost, which promotes constraint following.

5. The following equations describe the problem of Iterative Learning Control (ILC):

$$
\begin{aligned}
y_k(t) &= G_c(q)r(t) + G_c(q)u_k(t) \\
e_k(t) &= r(t) - y_k(t) \\
u_k(t) &= Q(q)[u_{k-1}(t) + L(q)e_{k-1}(t)]
\end{aligned}
$$

where $G_c(q)$ is the closed-loop transfer function of the system and q is the forward time shift operator.

a. Explain the principle of ILC and a draw a block diagram of the system. (1 p)

b. Give two examples of applications where ILC would be a suitable control strategy. (1 p)

c. Assume that $Q(q) = 1$ and that

$$G_C(q) = \frac{1}{q - 0.9}, \qquad L(q) = (\alpha q + 1)(q - 0.9)$$

Determine how to choose α in order to assure ILC stability and error convergence. (1 p)

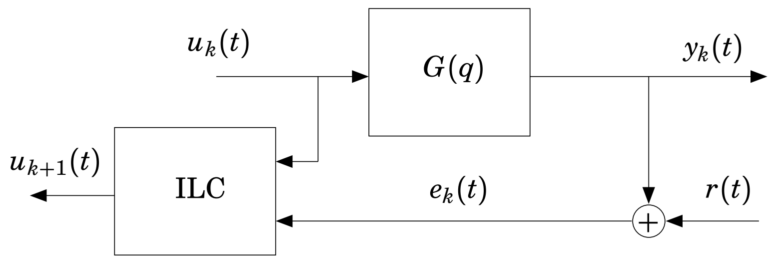
d. It is desired that the error dynamics of the ILC algorithm should be given by:

$$e_k(t) = H(q)e_{k-1}(t)$$

Assuming a model of the closed loop system $\hat{G}_c$ is available, and that $Q(q) = 1$, explain how to design the filter $L(q)$ in order to achieve the desired error dynamics. (1 p)

Solution

a. ILC is suitable for systems that repeatedly follow the same reference trajectory $r(t)$ over a finite time interval $[0, t_f]$. The strategy is based on collection of a data set and filtering operations upon the data. Non-causal filtering may be used since the filtering is performed offline. Denote the output and input of repetition k by $y_k(t)$ and $u_k(t)$. The control signal for repetition $k+1$ is then calculated by iterating from $u_k(t)$ with a filter $L(z)$.



Figur 3 Block diagram of ILC algorithm for Problem 5

b. As stated in a, the control problem should consist of repeting a task many times. Two examples of this is

- Trajectory optimization for fluid-filled containers on an assembly line
- A robot arm manufacturing machine parts

c. We obtain the following recursive expression for the tracking error

$$e_k(t) = [(1 - Q)(1 - G_C)]y_d(t) + [Q(1 - L \cdot G_C)]e_{k-1}(t)$$

and convergence will be achieved if

$$\mid 1 - L(e^{i\omega h}) \cdot G_C(e^{i\omega h}) \mid < \mid Q^{-1}(e^{i\omega h}) \mid$$

where $\omega h \in [-\pi, \pi]$ and h is the sampling time—*i.e.*, the Nyquist curve of $L(z)G_C(z)$ should be contained in a region in the complex plane given by a circle with radius one centered at $z = 1$. Simplification gives

$$\mid 1 - \alpha e^{i\omega h} - 1 \mid < 1, \qquad \omega h \in [-\pi, \pi]$$

from which the range of $\alpha \in [-1, 1]$ is determined.

d. Since a model of the closed loop system is available, model-based ILC may be used. By choosing:

$$L(q) = \hat{G}_c^{-1}(1 - H(q))$$

we get the error dynamics:

$$e_k(t) = [(1 - Q)(1 - G_C)]y_d(t) + [Q(1 - L \cdot G_C)]e_{k-1}(t)$$

6. Consider the stable, linear system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ k \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

k is an unknown parameter. The control law is given by $u = \theta u_c$, where $\theta = \theta(t)$ is the controller parameter and u_c is the reference signal. The system can be written as $y = kG(s)u$, that is, k is the feedforward gain for the system, where

$$G(s) = \frac{1}{s^2 + s + 1}$$

a. We want to determine the control parameter θ, such that the output follows the reference model $y_m = k_0 G(s) u_c$. Introduce the state error $\tilde{x} = x - x_m$, and the output error $e = y - y_m$. Let the desired θ be $\theta_0 = k_0/k$. Then the state-space equations for the the system from $(\theta - \theta_0)u_c$ to e are given by

$$\dot{\tilde{x}} = A\tilde{x} + B(\theta - \theta_0)u_c$$
$$e = C\tilde{x}$$

Use Lyapunov theory to derive a control law which guarantees that $\tilde{x}$ goes to zero. Motivate your choice of the controller. What extra knowledge do we need? (2 p)

Hint: Use the Lyapunov function

$$V = \frac{1}{2}\left(\tilde{x}^T P \tilde{x} + \frac{1}{\gamma}(\theta - \theta_0)^T(\theta - \theta_0)\right)$$

b. We can use the output error e instead of the state error $\tilde{x}$ in the update law for the parameter θ and guarantee that e goes to zero if $G(s)$ is SPR. Assume that $G(s)$ is SPR, and derive such an update law. (2 p)

c. Is the transfer function $G(s)$ SPR? Also check whether

$$G_1(s) = \frac{1}{s + 1}$$

is SPR.

(1 p)

Solution

a. Since the system

$$\dot{\tilde{x}} = A\tilde{x} + B(\theta - \theta_0)u_c$$

is asymptotically stable, there exist positive definite matrices P and Q such that

$$A^T P + PA = -Q.$$

Now choose $Q = I$, and let P be a symmetric matrix, i.e.

$$P = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}$$

The elements of P are obtained by solving the following linear system of equations:

$$\begin{pmatrix} 0 & -2 & 0 \\ 1 & -1 & -1 \\ 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}.$$

Solving this system gives $p_1 = 1.5$, $p_2 = 0.5$ and $p_3 = 1$.
Now we can choose the Lyapunov candidate

$$V = \frac{1}{2}\left(\tilde{x}^T P\tilde{x} + \frac{1}{\gamma}(\theta - \theta_0)^T(\theta - \theta_0)\right)$$

Its derivative is given by

$$\frac{dV}{dt} = -\frac{1}{2}\tilde{x}^T Q\tilde{x} + \frac{1}{\gamma}\left(\frac{d\theta}{dt} + \gamma\Psi^T P\tilde{x}\right),$$

where $\Psi = Bu_c = k\bar{B}u_c$, and

$$\bar{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Choosing the update law

$$\begin{aligned}\frac{d\theta}{dt} &= -\gamma\Psi^T P\tilde{x} \\ &= -\gamma u_c B^T P\tilde{x} \\ &= -\gamma k u_c \bar{B}^T P\tilde{x} \\ &= -\gamma' u_c \bar{B}^T P\tilde{x}\end{aligned}$$

gives that

$$\frac{dV}{dt} = -\frac{1}{2}\tilde{x}^T Q\tilde{x}$$

which is negative definite, and hence the state error $\tilde{x}$ will go to zero as t goes to infinity. Notice that since $\gamma' = \gamma k$, we need to know the sign of k.

b. If $G(s)$ (with realization A, $\bar{B}$, and C) is SPR, then there is – according to the Kalman-Yakubovich lemma – a Q such that the solution P to the Lyapunov equation fullfills

$$C = \bar{B}^T P.$$

If we use this equality in the above update law we obtain

$$\frac{d\theta}{dt} = -\gamma' u_c \bar{B}^T P \tilde{x} = -\gamma' u_c e.$$

c. We see that

$$\begin{aligned}
\mathrm{Re}(G(i\omega)) &= \mathrm{Re}\frac{1}{(i\omega)^2 + i\omega + 1} = \mathrm{Re}\frac{1}{-\omega^2 + i\omega + 1} \\
&= \mathrm{Re}\frac{1-\omega^2 - i\omega}{(1-\omega^2+i\omega)(1-\omega^2-i\omega)} = \mathrm{Re}\frac{1-\omega^2-i\omega}{(1-\omega^2)^2+\omega^2} \\
&= \frac{1-\omega^2}{(1-\omega^2)^2+\omega^2}
\end{aligned}$$

and is clearly negative when $|\omega|$ is greater than 1, which does not satisfy condition (iii) for positive realness .

$G_1(s)$ is SPR, since

(i) $G_1(s)$ has no poles in the RHP

(ii) $G_1(s)$ has no poles or zeros on the imaginary axis

(iii) Re $G_1(i\omega) = 1/(1+\omega^2) \geq 0$ for all ω

7. Determine the Kalman filter and derive the steady-state estimation covariance and filter gain for the system

$$\begin{aligned}
x_{k+1} &= 0.4x_k + v_k \\
y_k &= x_k + e_k
\end{aligned}$$

where v_k and e_k are zero-mean, uncorrelated white noise processes with variance 1. Compare the steady-state estimation covariance to that of x_k using the direct measurement as an estimate of $\bar{x}_k = y_k$ to predict x_k. Consider the time-invariant case only! (3 p)

Solution

The Kalman filter equations are

$$\begin{aligned}
\hat{x}_{k+1|k} &= 0.4\hat{x}_k + K_k(y_k - \hat{x}_k) \\
K_k &= \frac{0.4P_k}{1+P_k} \\
P_{k+1} &= 0.16P_k + 1 - \frac{0.16P_k^2}{1+P_k}, \quad P_k = E(\tilde{x}_k\tilde{x}_k)
\end{aligned}$$

If we denote the steady-state covariance by P_∞, the following holds

$$P_\infty = 0.16P_\infty + 1 - \frac{0.16P_\infty^2}{1+P_\infty} \to P_\infty^2 - 0.16P_\infty - 1 = 0$$

Solving for P_∞ and taking the positive solution gives $P_\infty \approx 2.81$. The corresponding steady-state filter gain is

$$K_\infty = \frac{0.4P_\infty}{1 + P_\infty} \approx 0.29$$

With the estimator $\bar{x}_k = y_k$, the prediction is $\bar{x}_{k+1} = 0.4y_k$. The estimation variance becomes

$$\begin{aligned} E\{(x_{k+1} - \hat{x}_{k+1})^2\} &= E\{(x_{k+1} - 0.4(x_k + e_k))^2\} = E\{(0.4x_k + v_k - 0.4(x_k + e_k))^2\} \\ &= E\{(v_k - 0.4e_k)^2\} = E\{v_k^2\} + 0.16E\{e_k^2\} \end{aligned}$$