1 Sampling

We start out with a continuous time system on state space form:

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$
$$y(t) = Cx(t). \tag{2}$$

If the system is given as a continuous time transfer function, you can easily rewrite it as one of several state space representations. (Search for canonical state space forms if you get stuck.)

To solve (1) we multiply both sides with the integrating factor e^{-At} to obtain

$$e^{-At}\dot{x}(t) = e^{-At}Ax(t) + e^{-At}Bu(t) \Leftrightarrow \tag{3}$$
$$e^{-At}\dot{x}(t) - e^{-At}Ax(t) = \frac{d}{dt}\left[e^{-At}x(t)\right] = e^{-At}Bu(t). \tag{4}$$

Hence

$$e^{-At}x(t) = e^{-At_k}x(t_k) + \int_{t_k}^{t} e^{-A\tau}Bu(\tau)d\tau, \tag{5}$$

by the Fundamental theorem of Calculus. Zero order hold sampling implies

$$u(\tau) = u(t_k) \;\; forall \tau \in]t_k, t_k + h], \tag{6}$$

and hence (5) can be rewritten

$$x(t_k + h) = e^{A(t_k+h-t_k)}x(t_k) + \int_{t_k}^{t_k+h} e^{A(t_k+h-\tau)}d\tau. \tag{7}$$

Introucing

$$h = t_{k+1} - t_k, \tag{8}$$
$$s = \tau - t_k, \tag{9}$$

we can rewrite (7)

$$x(t_k + h) = e^{Ah} + \int_0^h e^{As}dsBu(t_k). \tag{10}$$

Finally we introduce the constant matrices

$$\Phi_h = e^{Ah}, \tag{11}$$
$$\Gamma_h = \int_0^h e^{As}dsB, \tag{12}$$

and arrive at the sampled system

$$x(t_k + h) = \Phi_h x(t_k) + \Gamma_h u(t_k), \tag{13}$$
$$y(t_k) = Cx(t_k). \tag{14}$$