FRTN15 Predictive Control—Home Work 1
Signals and Systems

In this homework exercise we recapitulate theory for discrete time signals and systems in assignments 1-4. Recursive Least square estimation (RLS) is treated in assinment 5. The exercise also gives the oportunity to reherse (or learn) Matlab/Simulink.

E-mail your detailed and motivated solutions in pdf-format to FRTN15@control.lth.se. Attach any Matlab code or Simulink models you might have used.

1. Sample the (continuous time) double integrator

$$G_{y,u}(s) = \frac{1}{s^2}.$$

Provide the transfer function and a state space representation. Conduct the calculations

a. by hand, with a parametrized sample period h.

b. using matlab, with sample period $h = 1$.

c. How does the sampling period affect the result?

Solution

a. The observable canonical state space form of the continuous time system is given by

$$\begin{aligned}\frac{dx}{dt} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \quad (= Ax + Bu)\\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x \quad (= Cx).\end{aligned}$$

Hence, with sampling period h one obtains

$$\begin{aligned}x(kh+h) &= \Phi x(kh) + \Gamma u(kh)\\ y(kh) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x(kh)\end{aligned}$$

where

$$\Phi = e^{Ah} = I + Ah + A^2h^2/2 + \cdots = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}.$$

and

$$\Gamma = \int_0^h \begin{pmatrix} s \\ 1 \end{pmatrix} ds = \begin{pmatrix} \frac{h^2}{2} \\ h \end{pmatrix}.$$

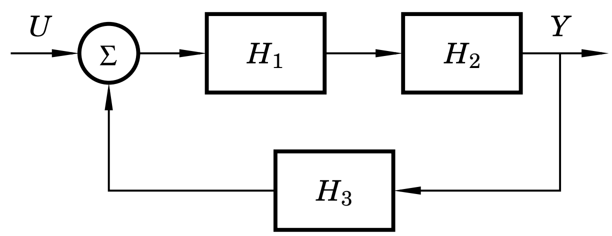
The transfer function representation is given by

$$H_{y,u}(z) = C(zI - \Phi)^{-1}\Gamma = \frac{h^2}{2}\frac{z+1}{(z-1)^2}.$$

b.
```
s=tf('s');
Gc=1/s^2;
h=1;
Gd=c2d(G,h)
[Phi,Gamma,C,D]=tf2ss(Gd.num{1},Gd.den{1})
```
If you are uncertain what a command does, type `help` followed by the command name in the Matlab prompt.

c. The sampled model behaves like its continuous time counterpart up to a higher frequency when sample period is decreased. However, decreasing sampling period leads to higher hardware demands and introduces numerical sensitivity when implemented on finite word lenght machines.

2. Give the transfer function $H(z)$ from u to y of the interconnection below, where $H_1(z) = z^2 + 3z + 2$, $H_2(z) = \frac{1}{z+2}$ and $H_3(z) = \frac{z}{z+1}$.



Solution

Simple algebra yields

$$H_1 H_2 (U - H_3 Y) = Y \Rightarrow Y = HU = \frac{H_1 H_2}{1 + H_1 H_2 H_3} U,$$

i.e.

$$H(z) = \frac{1+z}{1-z}.$$

3.

a. Show that the system H given by

$$x(kh+h) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} u(kh)$$
$$y(kh) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(kh)$$

is not asymptotically stable.

b. Stabilize H by means of linear state feedback. The resulting system should have all eigenvalues placed in the origin.

c. Placing all eigenvalues in the origin is refered to as deadbeat control. Give an interpretation, motivating this nomenclature. What potential drawbacks are there, compared to less aggressive eigenvalue placements?

Solution

a. A discrete time LTI is stable iff the spectrum of Φ is strictly contained within the unit disc. The system is hence not asymptotically stable. ($1 \in \mathrm{Sp}(\Phi)$.)

b. Let the linear state feedback be described by $u = -Kx$, i.e.

$$x(kh + h) = (\Phi - \Gamma K)x.$$

Inserting numerical values of K, Γ and solving

$$\mathrm{Sp}(\Phi - \Gamma K) = \{0, 0\}$$

for K yields

$$K = \begin{pmatrix} 1 & \frac{3}{2} \end{pmatrix}$$

In Matlab the `acker` command can be used to achieve this:

```
Phi=[1 1;0 1];
Gamma=[1/2 1]';
K=acker(Phi,Gamma,[0 0])
```

4. c. Show that for $X(z) = Z\{x(k)\}$

$$x(k) = \cos(\omega k)\theta(k) \Rightarrow X(z) = \frac{1 - z^{-1}\cos(\omega)}{1 - 2z^{-1}\cos(\omega) + z^{-2}}$$

where

$$\theta(k) = \begin{cases} 0 & , k < 0 \\ 1 & , k \geq 0 \end{cases}.$$

Solution

Euler's formula yields

$$\cos(\omega k) = \frac{e^{i\omega k} + e^{-i\omega k}}{2}.$$

The z-transform of $e_+(k) = e^{i\omega k}$ is given by

$$E_+(z) = \sum_{k=-\infty}^{\infty} e^{i\omega k} z^{-k} == \frac{z}{z - e^{i\omega}}.$$

Similarily, one obtains

$$e_-(z) \Longrightarrow E_-(z) = Z(e_-(k)) = \frac{z}{z - e^{-i\omega}}$$

(Here the choice between bi- and unilateral z-transform does not affect the result, since the sequence $x(k)$ is identically 0 for $k < 0$.)

5.

a. Use Simulink to demonstrate RLS identification $\Theta = [a\ b]^T$ of $y_k = ay_{k-1} + bu_{k-1} + w_k$, where w_k is a normal white noise process.

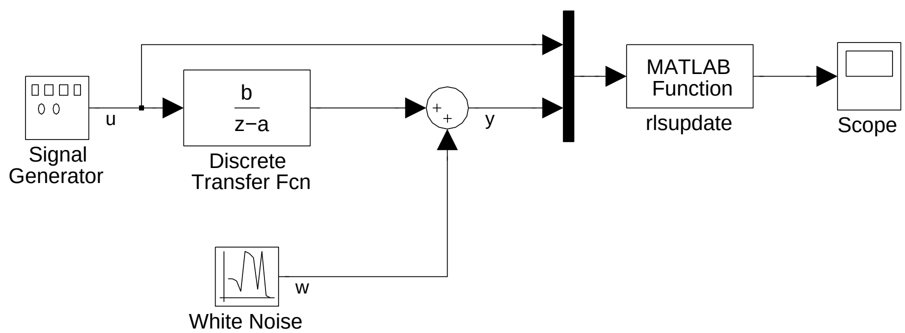
Hint: Use the 'Matlab function' block in Simulink.
Type 'help persistent. at the Matlab prompt.

b. Comment on the choice of input signal and how it affects the result.

c. Assume a is time varying and that the highest frequency in $a(t)$ is known. How can the method be modified to identify a?

Solution

a. This can be done in many ways. One possibility is to use the simulink model depicted below: The following settings have been applied:



- 'Configuration Parameters' from the 'Simulation menu': 'Stop time' under 'Simulation time' changed to '300', 'Solver' under 'Solver options' changed to 'Discrete (no continuous states)'
- 'Signal generator': 'Wave form' changed to 'square', 'Frequency' changed to '1/uP'
- 'MATLAB Function': 'Matlab functio' changed to 'rlsupdate'

System parameters were defined as follows:

```
h = 1.0;    % sample period [s]
a = 0.9;
b = 0.1;
uA = 1.0;   % amplitude of input (square)
uP = 75*2;  % period of input (square) [s]
```

The RLS update function was defined:

```
function thetaHat = rlsupdate(uy)
persistent P theta phi
if isempty(P)
    P = eye(2);
    theta = [0 0]';
    phi = [0 0]';
```

```
end
u = uy(1);
y = uy(2);
P = P-(P*phi*phi'*P)/(1+phi'*P*phi);
e = y - phi'*theta;
theta = theta + P*phi*e;
thetaHat = theta;
phi = [y u]';
return
```

b. We need a signal which excites the system dynamics enough. Since two parameters are to be identified, it is sound to apply an input with persistent excitation index ≥ 2. However, due to the AR part of the system, it might be enough with persisten excitation index one, if the signal to noise ratio is high.

c. Augment it with exponential forgetting.