LUNDS TEKNISKA HÖGSKOLA
Lunds universitet

Institutionen för
REGLERTEKNIK

FRTN15 Predictive Control

Final Exam October 25, 2008, 8-13

General Instructions

This is an open book exam. You may use any book you want. However, no previous exam sheets or solution manuals are allowed. The exam consists of 6 problems to be solved. Your solutions and answers to the problems should be well motivated. The credit for each problem is indicated in the problem. The total number of credits is 25 points. Preliminary grade limits are:

Grade 3:	12 points
Grade 4:	17 points
Grade 5:	22 points

Results

The results of the exam will be posted at the latest November 1 on the notice board on the first floor of the M-building and they will also be available on the course home page.

Do you accept publication of your grading result on our local web page? (Godkänner du publicering av resultatet på vår lokala hemsida?)

1. The following system is to be controlled using Model Predictive Control.

$$
\begin{aligned}
x(k+1) &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k) \\
y(k) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x(k)
\end{aligned}
$$

where $y(k)$ is available for measurement. The controller should fulfill $y(k) = r(k)$, where $r(k)$ is a reference signal, and respect the constraints

$$
\begin{aligned}
x_{min} \leq \; & x_2(k) \; \leq x_{max} \\
u_{min} \leq \; & u(k) \; \leq u_{max}
\end{aligned}
$$

a. Determine the additional output signals z and z_c which correspond to controlled and constrained outputs respectively, i.e. determine C_z and C_c so that the system can be written on the following form

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) \\
y(k) &= Cx(k) \\
z(k) &= C_z x(k) \\
z_c(k) &= C_c x(k)
\end{aligned}
$$

(1 p)

b. For the numerical values $u_{min} = x_{min} = -2$ and $u_{max} = x_{max} = 2$, and given that $x(k) = (0 \quad 1)^T$ and $u(k-1) = 0.5$, what restrictions do we have on $\Delta u(k) = u(k) - u(k-1)$? (2 p)

c. What computational problems can arise if the plant is operated near the constraint on x_2 and noise disturbances are affecting the states? How can the MPC problem be modified to limit these problems? (1 p)

d. Assume that a constant disturbance is acting on the process input. How can the state space model be extended to include this? (1 p)

Solution

a. We have a constraint on x_2, i.e. $z_c(k) = x_2(k)$, $C_c = (0 \; 1)$. We have a set-point for x_1 which gives $z(k) = x_1(k)$ $C_z = (1 \; 0)$.

b. From the constraint $-2 \leq u(k) \leq 2$ and the relation $\Delta u(k) = u(k) - u(k-1)$ we get $-2 + 0.5 = -2.5 \leq \Delta u(k) \leq 1.5 = 2 - 0.5$.

The state space-model gives $x_2(k+1) = x_2(k) + u(k)$ which can be expanded to $x_2(k+1) = x_2(k) + u(k-1) + \Delta u(k) = 1 + 0.5 + \Delta u(k)$. From the constraint $-2 \leq x_2 \leq 2$ we get $-2 - 1.5 = -3.5 \leq x_2(k) \leq 0.5 = 2 - 1.5$.

The lower bound on $\Delta u(k)$ is given by the constraint on $u(k)$ and the upper bound is given by the constraint on $x_2(k)$, we get

$$-2.5 \leq \Delta u(k) \leq 0.5$$

c. If a disturbance is acting on the system the constraint on x_2 might be violated, which in turn may cause problems for the solver. One way of limiting this problem is to soften the constraint on x_2. This can be done by replacing $x_{min} \le x_2 \le x_{max}$ with

$$x_{min} - \epsilon (V_k^x)_{min} \le x_2(k) \le x_{max} + \epsilon (V_k^x)_{max}$$

where ϵ is the slack variable and $(V_k^x)_{min}$ and $(V_k^x)_{max}$ are relaxation vectors and adding an extra term $\rho_\epsilon \epsilon^2$ to the cost function. The parameter ρ_ϵ then determines the amount of softening.

d. With a constant disturbance d on the input the system can be written as

$$\begin{aligned} x(k+1) &= Ax(k) + B(u(k) + d(k)) \\ y(k) &= Cx(k) \end{aligned}$$

Introducing the extended state vector $x_e(k) = (x(k)\ d(k))^T$ we can write

$$\begin{aligned} x_e(k+1) &= \begin{pmatrix} A & B \\ 0 & 1 \end{pmatrix} x_e(k) + \begin{pmatrix} B \\ 0 \end{pmatrix} u(k) \\ y(k) &= \begin{pmatrix} C & 0 \end{pmatrix} x_e(k) \end{aligned}$$

2. We want to estimate the parameters a and b using input-output data for a system with transfer function $H(z)$ given by

$$H(z) = \frac{b}{z + a}$$

The parameters are likely to be time-varying, so Recursive Least Squares (RLS) estimation with forgetting is employed.

a. State the equations for the RLS-algorithm with forgetting and explain its operation. Also, write the model on regressor form. (2 p)

b. What problems might arise when a small value of λ is used? (1 p)

c. Figure 1 shows the results when using the four configurations i-iv of λ and κ, where $P_0 = \kappa \cdot I$.

$$\begin{aligned} \text{i}: &\quad \lambda = 0.998, \quad \kappa = 10^3 \\ \text{ii}: &\quad \lambda = 1, \quad \kappa = 10^3 \\ \text{iii}: &\quad \lambda = 0.998, \quad \kappa = 1 \\ \text{iv}: &\quad \lambda = 1, \quad \kappa = 10^{-2} \end{aligned}$$

The initial estimates were 0.5 for both parameters. The correct values are $a = 0.1$ and $b = 0.3$. At $t = 50$, a changes to 0.2. Determine which result A-D that corresponds to which configuration i-iv. Motivate your answer. (2 p)

d. Assume that some input-output data are available off-line. Explain how this can be used to improve the initial behaviour of the on-line algorithm. (1 p)

Solution

a. The input-output data are connected via

$$y(k+1) + ay(k) = bu(k)$$

where $y(k)$ is the output and $u(k)$ is the input. To write the model on regressor form introduce

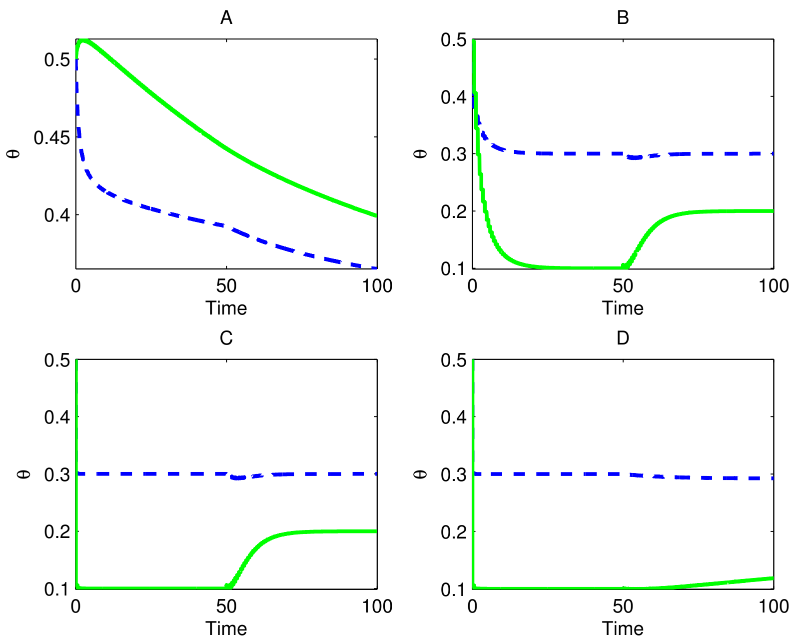
$$\phi(k) = \begin{pmatrix} -y(k) \\ u(k) \end{pmatrix}, \quad \theta = \begin{pmatrix} a \\ b \end{pmatrix}$$

The regressor form is then $y(k+1) = \phi^T(k)\theta$.
The recursive algorithm with forgetting is given by

$$\begin{aligned} \hat{\theta}_k &= \hat{\theta}_{k-1} + P_k \phi_k \epsilon_k \\ \epsilon_k &= y_k - \phi_k^T \hat{\theta}_{k-1} \\ P_k &= \frac{1}{\lambda}\left(P_{k-1} - \frac{P_{k-1}\phi_k\phi_k^T P_{k-1}}{\lambda + \phi_k^T P_{k-1}\phi_k}\right) \end{aligned}$$

The first equation is the estimate update. The second equation is the prediction error using $\hat{\theta}_{k-1}$. The third equation updates the covariance estimate P_k. When $0 < \lambda < 1$ the algorithm emphasizes fitting of recent data and reduces the influence of old data.

b. One drawback with using a small λ is that the noise sensitivity increases. Another problem is that the P_k-matrix may increase with k if $P_{k-1}\phi_k$ is small, sometimes referred to as “P-matrix explosion”.



Figur 1 Responses to the parameter choices i-iv in Problem 2.

c. Subplots C and D have the fastest initial transients, which indicates that they correspond to a large value of κ. Subplot C has faster adaption to the new parameter values than D, which indicates a lower value of λ. Subplot A has the slowest initial transient, which indicates a small value of κ while subplot B has a medium fast initial transient and also displays adaption to the new parameters similar to that in subplot C. Taking this together we get

$$\begin{aligned} A &- iv \\ B &- iii \\ C &- i \\ D &- ii \end{aligned}$$

d. The off-line data can be used to obtain initial estimates of θ_0 and P_0 to reduce initial transients.

3.

a. Explain the basic principles in Model Reference Adaptive Control (MRAC). What is the difference between *direct* and *indirect* MRAC? What design choices are needed? What information about the plant is needed? (2 p)

b. Consider the system $H(q)$ given by

$$H(q) = \frac{b_0 q + b_1}{q^2 + a_1 q + a_2}$$

where b_0 and b_1 are such that the process zero is stable. Show how a direct MRAC on the form

$$R(q)u(t) = T(q)u_c(t) - S(q)y(t)$$

is obtained so that the closed-loop transfer function from u_c to y is given by

$$H_m(q) = \frac{b_{m1} q + b_{m2}}{q^2 + a_{m1} q + a_{m2}}$$

What are the orders of the polynomials $R(q)$, $S(q)$, and $T(q)$? Assume that the process zero is cancelled. (2 p)

Solution

a. The objective of Model Reference Control is to design a controller such that the closed loop dynamics of controller and plant match a specified desired model.

If the controller is on the form

$$R(z)u(k) = -S(z)y(k) + T(z)u_c(k)$$

where $y(k)$ is the plant output, $u(k)$ is the control signal, and $u_c(k)$ is the command signal we want to choose the polynomials $R(z)$, $S(z)$, and $T(z)$ so that a system with transfer function $G(z)$ has specified closed-loop dynamics $G_m(z)$.

When $G(z)$ is unknown, the controller must be made adaptive. In the *direct* approach, the polynomials are determined directly from input-output data (see 3. b.). In the case of *indirect* MRAC, the plant parameters are estimated and the polynomials are then calculated.

The design choices include:

- Choosing a reference model G_m
- Choosing whether to cancel any process zeros or not
- Choosing an observer polynomial

Typically, we need to know the order of the plant, but not the specific parameters.

b. We introduce polynomials A, B, A_m, B_m according to

$$B = b_0 q + b_1, \quad A = q^2 + a_1 q + a_2, \quad B_m = b_{m1} q + b_{m2}, \quad A_m = q^2 + a_{1m} q + a_{2m}$$

so that we have $Ay(t) = Bu(t)$ and $A_m y_m(t) = Bu_c(t)$.

Next we factor B as $B = B^+ B^-$ where B^+ is subject to cancellation.

$$B^+ = q + \frac{b_1}{b_0}, \quad B^- = b_0$$

We begin by letting the Diophantine equation $AR + BS = A_m A_o B^+$ act on the output $y(t)$.

$$ARy(t) + BSy(t) = A_m A_o B^+ y(t)$$

Inserting $Ay(t) = Bu(t)$ we have

$$BRu(t) + BSy(t) = A_m A_o B^+ y(t)$$

Cancelling B^+ we get

$$b_0 Ru(t) + b_0 Sy(t) = A_m A_o y(t)$$

This can be re-arranged as a regression model

$$y(t) = R\left(\frac{b_0}{A_m A_o} u(t)\right) + S\left(\frac{b_0}{A_m A_o} y(t)\right)$$

If b_0 is known, we can estimate R and S directly using the expressions in parantheses as regressors and choose $T = (B_m A_o)/b_0$.

When b_0 is unknown, one possibility is to rewrite the regression model so that the products $R' = b_0 R$ and $S' = b_0 S$ are estimated, and then identify b_0 as the common factor. We could also identify b_0 separately.

4. Consider the following system

$$\begin{aligned} x_{k+1} &= a x_k + v_k \\ y_k &= x_k + e_k \end{aligned}$$

where v and e are white-noise processes with zero mean and the covariances

$$\begin{aligned}
\mathbf{E}\{v_j v_k\} &= r_1 \delta_{jk} \\
\mathbf{E}\{v_j e_k\} &= 0 \\
\mathbf{E}\{e_j e_k\} &= r_2 \delta_{jk}
\end{aligned}$$

a. Determine the Kalman filter for the system. (1 p)

b. What are the steady-state filter gain and the estimation covariance? (1 p)

c. Determine the filter gain and estimation covariance in steady-state when $r_1 >> r_2$. Comment on the result. (1 p)

Solution

a. The Kalman filter equations are given in Table 7.1 in the text-book. For the system in question they reduce to

$$\begin{aligned}
P_{k+1|k} &= a^2 P_{k|k-1} + r_1 - \frac{a^2 P_{k|k-1}^2}{r_2 + P_{k|k-1}} \\
K_k &= \frac{a P_{k|k-1}}{r_2 + P_{k|k-1}} \\
\hat{x}_{k+1|k} &= a \hat{x}_{k|k-1} + B u_k + K_k (x_k - \hat{x}_{k|k-1})
\end{aligned}$$

b. In steady-state the covariance matrix P_∞ and the filter gain K_∞ are given by the equations

$$\begin{aligned}
P_\infty &= a^2 P_\infty + r_1 - \frac{a^2 P_\infty^2}{r_2 + P_\infty} \\
K_\infty &= \frac{a P_\infty}{r_2 + P_\infty}
\end{aligned}$$

c. When $r_1 >> r_2$ we can approximate the equation for P_∞ by

$$P_\infty \approx a^2 P_\infty + r_1 - \frac{a^2 P_\infty^2}{P_\infty} = r_1$$

which yields

$$K_\infty \approx \frac{a r_1}{r_1} = a$$

This means that the observer poles are placed in the origin, we have obtained a dead-beat observer which trusts the measurements much more than the model output.

5.

a. Calculate a one-step-ahead optimal predictor for the system

$$y_{k+1} = 0.8 y_k - 0.5 y_{k-1} + w_{k+1} + 0.4 w_k$$

where w is a stochastic process with zero mean and variance $\mathbf{E}\{w_k w_j^T\} = \sigma_w^2 \delta_{kj}$. Also determine the prediction covariance. (2 p)

b. Determine a one-step-ahead minimum-variance controller for the system

$$y_{k+1} = 0.8y_k - 0.5y_{k-1} + u_k + 0.6u_{k-1} + w_{k+1} + 0.4w_k$$

Also determine the output variance under closed-loop minimum variance control. (2 p)

Solution

a. We write the system as

$$y_k = \frac{C(z^{-1})}{A(z^{-1})} w_k$$

where

$$A(z^{-1}) = 1 - 0.8z^{-1} + 0.5z^{-2}, \quad C(z^{1-}) = 1 + 0.4z^{-1}$$

From the diophantine equation

$$C(z^{-1}) = A(z^{-1})F(z^{-1}) + z^{-1}G(z^{-1})$$

we see that we can write

$$y_{k+1} = F(z^{-1})w_{k+1} + \frac{G(z^{-1})}{A(z^{-1})} w_k$$

As w_{k+1} is unknown at time k the optimal prediction of y_{k+1} is

$$\hat{y}_{k+1} = \frac{G(z^{-1})}{A(z^{-1})} w_k = \frac{G(z^{-1})}{C(z^{-1})} y_k$$

where the last equality is found by solving the system equation for w_k. With $F(z^{-1})$ of order $d-1$ where $d = 1$ is the delay of the system and $G(z^{-1})$ of order $n-1$ where n is the order of the system, gives

$$F = f_0$$
$$G = g_0 + g_1 z^{-1}$$

The coefficients are found by comparing powers of z^{-1} in the diophantine equation, yielding $f_0 = 1$, $g_0 = 1.2$, $g_1 = -0.5$. The prediction covariance is $E\{(\hat{y}_{k+1} - y_{k+1)^2}\} = f_0^2 \sigma_w^2 = \sigma_w^2$.

b. Using the same notation and diophantine equation as in a., and introducing $B(z^{-1}) = 1 + 0.6z^{-1}$ we obtain

$$y_{k+1} = F(z^{-1})w_{k+1} + \frac{B(z^{-1})}{A(z^{-1})} u_k + \frac{G(z^{-1})}{A(z^{-1})} w_k$$

Solving the system equation for w_k and rearranging yields

$$y_{k+1} = F(z^{-1})w_{k+1} + \frac{B(z^{-1})F(z^{-1})}{C(z^{-1})} u_k + \frac{G(z^{-1})}{C(z^{-1})} y_k$$

The minimum variance controller is obtained by choosing u_k so that the last two terms cancel out.

$$u_k = -\frac{G(z^{-1})}{B(z^{-1})F(z^{-1})} y_k$$

The output variance is the same as the prediction covariance in a.

6. An Iterative Learning Control (ILC) strategy for the system

$$y_k(t) = G(q)u_k(t)$$

is given by

$$\begin{aligned} e_k(t) &= r(t) - y_k(t) \\ u_{k+1}(t) &= u_k(t) + L(q)e_k(t) \end{aligned}$$

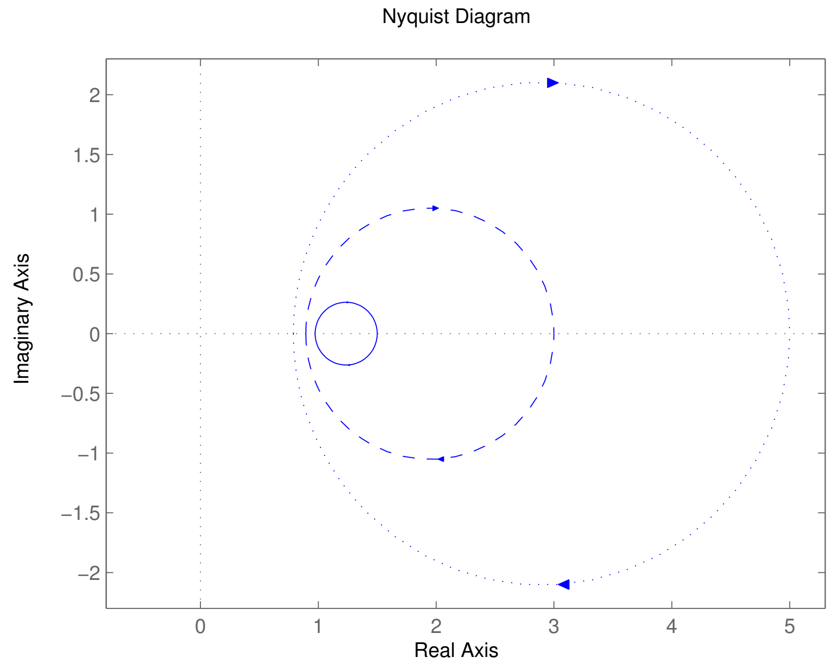
a. Explain the principle of ILC and a draw a block diagram of the system. (1 p)

b. Give two examples of practical situations where ILC would be a suitable control strategy. (1 p)

c. Figure 2 shows the nyquist plot for the three choices of $L(q)$:

$$\begin{aligned} L(q) &= q + 0.5 \quad \textit{(dotted)} \\ L(q) &= q + 0.7 \quad \textit{(dashed)} \\ L(q) &= q + 0.85 \quad \textit{(solid)} \end{aligned}$$

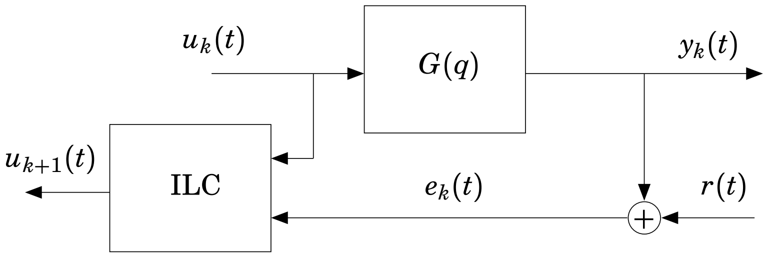
For which choice of $L(q)$ will the control error converge? (1 p)



Figur 2 Nyquist plots for G(q)L(q) in Problem 6.

Solution

a. ILC is suitable for systems that repeatedly follow the same reference trajectory $r(t)$ over a finite time interval $[0, t_f]$. Denote the output and input of repetition k by $y_k(t)$ and $u_k(t)$. The control signal for repetition $k+1$ is then calculated by iterating from $u_k(t)$ with a filter $L(q)$.



Figur 3 Block diagram of ILC algorithm for Problem 6.

b. As stated in a, the control problem should consist of repeting a task many times. Two examples of this is

- Trajectory optimization for fluid-filled containers on an assembly line
- A robot arm manufacturing machine parts

c. A sufficient condition for stability is that

$$\sup_{\omega h \in [-\pi,\pi]} ||I_p - G(z)L(z)||_{|z=e^{i\omega h}} < 1$$

i.e. the Nyquist curve of $G(z)L(z)$ should be contained in a circle with radius one centered at $z = 1$. We see that only $L(q) = q+0.85$ fulfills this condition.