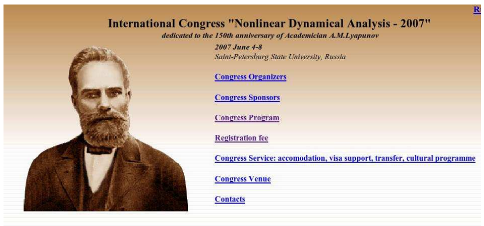
FRTN15 Lecture 12—Stability

Outline

- Lyapunov Stability
- Strictly Positive Realness (SPR)
- Kalman-Yakubovich-Popov Lemma
- Passivity
- Gain Adaptation
- Stability of MRAC

A. M. Lyapunov (1857–1918)



Master thesis “On the stability of ellipsoidal forms of equilibrium of rotating fluids,” St. Petersburg University, 1884.

Doctoral thesis “The general problem of the stability of motion,” 1892.

Main idea

Lyapunov formalized the idea:

If the total energy is dissipated, then the system must be stable.

Main benefit: By looking at **how** an energy-like function **V** (a so called *Lyapunov function*) **changes over time**, we might **conclude** that a system is stable or asymptotically stable **without solving** the nonlinear differential equation.

Main question: **How to find** a Lyapunov function?

Examples

Start with a Lyapunov *candidate* V to measure e.g.,

- "size"[1] of state and/or output error,
- "size" of deviation from true parameters,
- energy difference from desired equilibrium,
- weighted combination of above
- ...

Example of common choice in adaptive control

$$V = \frac{1}{2}\left(e^2 + \gamma_a \tilde{a}^2 + \gamma_b \tilde{b}^2\right)$$

(here weighted sum of output error and parameter errors)

[1] Often a magnitude measure or (squared) norm like $|e|_2^2$, ...

Analysis: Check if V is decreasing with time

- Continuous time: $\dfrac{dV}{dt} < 0$
- Discrete time: $V(k+1) - V(k) < 0$

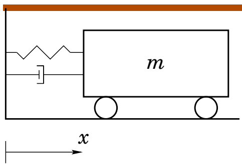
Synthesis: Choose e.g. control law and/or parameter update law to satisfy $\dot{V} \leq 0$

$$\frac{dV}{dt} = e\dot{e} + \gamma_a \tilde{a}\dot{\tilde{a}} + \gamma_b \tilde{b}\dot{\tilde{b}} =$$
$$= \tilde{x}(-a\tilde{x} - \tilde{a}\tilde{x} + \tilde{b}u) + \gamma_a \tilde{a}\dot{\tilde{a}} + \gamma_b \tilde{b}\dot{\tilde{b}} = ...$$

If a is constant and $\tilde{a} = a - \hat{a}$ then $\dot{\tilde{a}} = -\dot{\hat{a}}$.

Choose update law $\dfrac{d\hat{a}}{dt}$ in a "good way" to influence $\dfrac{dV}{dt}$.
(more on this later...)

A Motivating Example



$$m\ddot{x} = -\underbrace{b\dot{x}|\dot{x}|}_{\text{damping}} - \underbrace{k_0 x - k_1 x^3}_{\text{spring}}$$
$$b, k_0, k_1 > 0$$

Total energy = kinetic + pot. energy: $V = \frac{mv^2}{2} + \int_0^x F_{spring}\, ds \;\Rightarrow$

$$V(x,\dot{x}) = m\dot{x}^2/2 + k_0 x^2/2 + k_1 x^4/4 > 0, \qquad V(0,0) = 0$$

$$\frac{d}{dt}V(x,\dot{x}) = m\ddot{x}\dot{x} + k_0 x\dot{x} + k_1 x^3\dot{x} = \{\text{plug in system dynamics}^2\}$$
$$= -b|\dot{x}|^3 \quad < 0, \text{ for } \dot{x} \neq 0$$

What does this mean?

[2] Also referred to evaluate “along system trajectories”.

Stability Definitions

An equilibrium point $x = 0$ of $\dot{x} = f(x)$ is

locally stable, if for every $R > 0$ there exists $r > 0$, such that

$$\|x(0)\| < r \quad \Rightarrow \quad \|x(t)\| < R, \quad t \geq 0$$

locally asymptotically stable, if locally stable and

$$\|x(0)\| < r \quad \Rightarrow \quad \lim_{t\to\infty} x(t) = 0$$

globally asymptotically stable, if asymptotically stable for all $x(0) \in \mathbf{R}^n$.

Lyapunov Theorem for Local Stability

Theorem Let $\dot{x} = f(x)$, $f(0) = 0$, and $0 \in \Omega \subset \mathbf{R}^n$. Assume that $V : \Omega \to \mathbf{R}$ is a C^1 function. If

(1) $V(0) = 0$
(2) $V(x) > 0$, for all $x \in \Omega$, $x \neq 0$
(3) $\frac{d}{dt}V(x) \leq 0$ along all trajectories of **the system** in Ω

then $x = 0$ is locally stable. Furthermore, if also

(4) $\frac{d}{dt}V(x) < 0$ for all $x \in \Omega$, $x \neq 0$

then $x = 0$ is locally asymptotically stable.

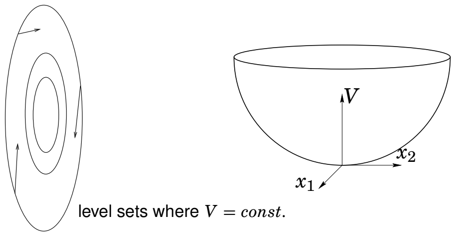
Lyapunov Functions (≈ Energy Functions)

A function V that fulfills (1)–(3) is called a *Lyapunov function*.

Condition (3) means that V is non-increasing along all trajectories in Ω:

$$\dot{V}(x) = \frac{d}{dt}V(x) = \frac{\partial V}{\partial x} \cdot \dot{x} = \frac{\partial V}{\partial x} \cdot f(x) \le 0$$

where $\frac{\partial V}{\partial x} = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} & \dots & \frac{\partial V}{\partial x_n} \end{bmatrix}$



Conservation and Dissipation

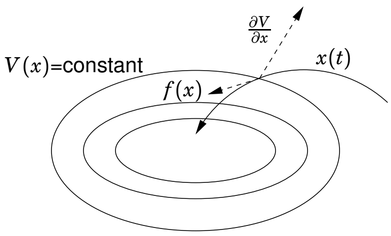
Conservation of energy: $\dot{V}(x) = \frac{\partial V}{\partial x} f(x) = 0$, i.e. the vector field $f(x)$ is everywhere orthogonal to the normal $\frac{\partial V}{\partial x}$ to the level surface $V(x) = c$.

Example: Total energy of a lossless mechanical system or total fluid in a closed system.

Dissipation of energy: $\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \le 0$, i.e. the vector field $f(x)$ and the normal $\frac{\partial V}{\partial x}$ to the level surface $V(x) = c$ make an obtuse angle (Sw. "trubbig vinkel").

Example: Total energy of a mechanical system with damping or total fluid in a system that leaks.

Geometric interpretation



Vector field points into sublevel sets

Trajectories can only go to lower values of $V(x)$

Boundedness:

For an trajectory $x(t)$

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau)d\tau \le V(x(0))$$

which means that the whole trajectory lies in the set

$$\{z \mid V(z) \le V(x(0))\}$$

For stability it is thus important that the sublevel sets $\{z \mid V(z) \le c)\}$ are locally bounded.

Lyapunov Theorem for Global Asymptotic Stability

Theorem Let $\dot{x} = f(x)$ and $f(0) = 0$.
If there exists a $\mathbb{C}^1$ function $V : \mathbf{R}^n \to \mathbf{R}$ such that

(1) $V(0) = 0$
(2) $V(x) > 0$, for all $x \ne 0$
(3) $\dot{V}(x) < 0$ for all $x \ne 0$
(4) $V(x) \to \infty$ as $\|x\| \to \infty$

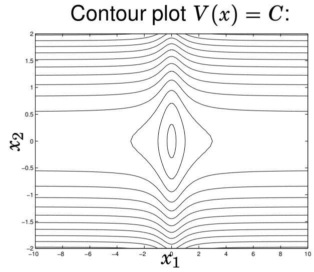
then $x = 0$ is globally asymptotically stable.

Radial Unboundedness is Necessary

If the condition $V(x) \to \infty$ as $\|x\| \to \infty$ is not fulfilled, then global stability cannot be guaranteed.

Example Assume $V(x) = x_1^2/(1 + x_1^2) + x_2^2$ is a Lyapunov function for a system. Can have $\|x\| \to \infty$ even if $\dot{V}(x) < 0$.

Contour plot $V(x) = C$:



Example:

$$\dot{x}_1 = \frac{-6x_1}{(1 + x_1^2)^2} + 2x_2$$
$$\dot{x}_2 = \frac{-2(x_1 + x_2)}{(1 + x_1^2)^2}$$

Example- Lyapunov fcn for linear system

$$\dot{x} = Ax \quad = \begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (1)$$

Eigenvalues of A : $\{-1, -3\} \Rightarrow$ (global) asymptotic stability.

Find a quadratic Lyapunov function

$$V(x) = x^T P x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad P = P^T > 0$$

for the system (1).

Solve the Lyapunov equation $A^T P + PA = -Q$. Take any $Q = Q^T > 0$, say $Q = I_{2\times 2}$.

Example cont'd

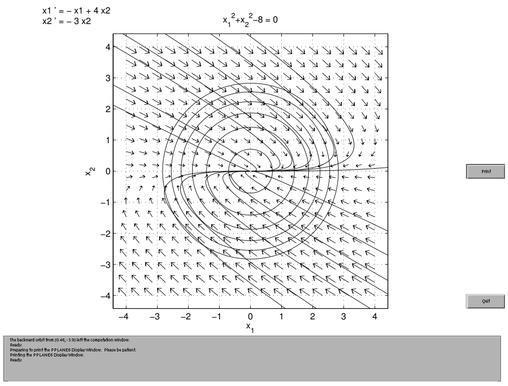
$$A^T P + PA = -I$$

$$\begin{bmatrix} -1 & 0 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} -2p_{11} & -4p_{12} + 4p_{11} \\ -4p_{12} + 4p_{11} & 8p_{12} - 6p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad (2)$$

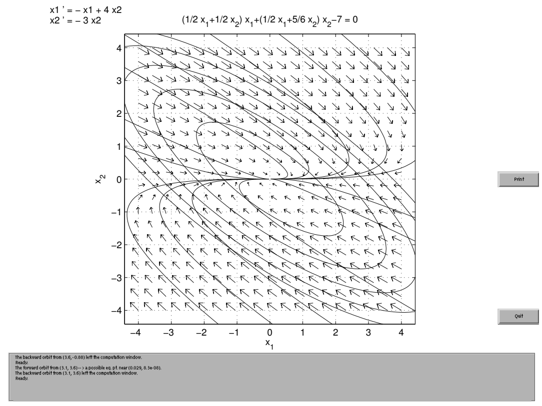
Solving for p_{11}, p_{12} and p_{22} gives

$$2p_{11} = -1$$
$$-4p_{12} + 4p_{11} = 0$$
$$8p_{12} - 6p_{22} = -1$$

$$\Longrightarrow \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 5/6 \end{bmatrix} > 0$$



Phase plot showing that
$V = \frac{1}{2}(x_1^2 + x_2^2) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ does NOT work.



Phase plot with level curves $x^T P x = constant$ for P found in example.

Somewhat Stronger Assumptions

Theorem: Let $\dot{x} = f(x)$ and $f(0) = 0$. If there exists a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ such that

(1) $V(0) = 0$
(2) $V(x) > 0$ for all $x \neq 0$
(3) $\dot{V}(x) \leq -\alpha V(x)$ for all x
(4) The sublevel sets $\{x | V(x) \leq c\}$ are bounded for all $c \geq 0$

then $x = 0$ is globally **exponentially** stable.

Proof Idea

Assume $x(t) \neq 0$ (otherwise we have $x(\tau) = 0$ for all $\tau > t$). Then

$$\frac{\dot{V}(x)}{V(x)} \leq -\alpha$$

Integrating from 0 to t gives

$$\log V(x(t)) - \log V(x(0) \leq -\alpha t \;\Rightarrow\; V(x(t)) \leq e^{-\alpha t} V(x(0))$$

Hence, $V(x(t)) \to 0$, $t \to \infty$.

Using the properties of V it follows that $x(t) \to 0$, $t \to \infty$.

Preliminaries

Definition (Strictly Positive Realness (SPR))

A proper rational transfer function matrix $H(s)$ is positive real if

- All elements of $H(s)$ are analytic for Re$[s] > 0$;
- Any pure imaginary pole of any element of $H(s)$ is a simple pole and the associated residue matrix of $H(s)$ is positive definite Hermitian;
- For all real ω for which $i\omega$ is not a pole of any element of $H(s)$, the matrix $H(i\omega) + H^T(-i\omega)$ is positive definite and strictly positive real (SPR) if $H(s - \varepsilon)$ is positive real for some $\varepsilon > 0$.

[Kalman-Yakubovich-Popov Lemma

Lemma (Kalman-Yakubovich-Popov [**?**, **?**, **?**])

Let $G_0(s) = C(sI - A)^{-1}B + D$ be a $m \times m$ transfer function where A is Hurwitzian, (A, B) is controllable and (A, C) is observable. Then, $G_0(s)$ is strictly positive real if and only if there exist a positive symmetric matrix P, matrices W_1, W_2 and a positive constant ϵ such that

$$\begin{aligned} PA + A^T P &= -W_1 W_1^T - \epsilon P, \\ PB - C^T &= -W_1 W_2^T, \quad D + D^T = W_2 W_2^T \end{aligned}$$

□

Lyapunov revisited

Original idea: "Energy is decreasing"

$$\begin{aligned} &\dot{x} = f(x), \qquad x(0) = x_0 \\ &V(x(T)) - V(x(0)) \leq 0 \\ &(+\text{some other conditions on } V) \end{aligned}$$

New idea: "Increase in stored energy $\leq$ added energy"

$$\begin{aligned} &\dot{x} = f(x, u), \qquad x(0) = x_0 \\ &y = h(x) \end{aligned}$$

$$V(x(T)) - V(x(0)) \leq \int_0^T \underbrace{\varphi(y, u)}_{\text{external power}} \; dt \qquad (3)$$

Passivity

Consider a system

$$\begin{aligned} \dot{x} &= f(x, u), \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^p, \\ y &= h(x, u), \qquad y \in \mathbb{R}^p \end{aligned}$$

where $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is locally Lipschitz, $h : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$ is continuous with $f(0, 0) = 0$, $h(0, 0) = 0$.

The system is said to be passive if there exists a continuously differentiable positive semidefinite function $V(x)$—the storage function—such that

$$u^T y \geq \dot{V} = \frac{\partial V}{\partial x} f(x, u), \forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^p$$

Passivity

The passive system is said to be

- lossless if $u^T y = \dot{V}$;
- strictly passive if $u^T y \geq \dot{V} + \varphi(x) > 0$ for some positive definite function φ;
- input strictly passive if $u^T y \geq \dot{V} + u^T \psi(u) > 0$ and $u^T \psi(u) > 0, \ \forall u \neq 0$;
- output strictly passive if $u^T y \geq \dot{V} + y^T \rho(y) > 0$ and $y^T \rho(y) > 0 \ \forall y \neq 0$;

if the inequality holds for all (x, u).

Dissipativity

Definition (Dissipativity)

A dynamical system is

$$\begin{aligned} \dot{x} &= f(x,u), \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^p, \\ y &= h(x,u), \qquad y \in \mathbb{R}^p \end{aligned}$$

is said to be dissipative with respect to a supply rate $w(u, y)$ if there exists a positive definite storage function $V(x)$ such that $\dot{V} \leq w$

Lyapunov vs. Passivity

Storage function is a generalization of Lyapunov function

Lyapunov idea: "Energy is decreasing"

$$\dot{V} \leq 0$$

Passivity idea: "Increase in stored energy $\leq$ Added energy"

$$\dot{V} \leq u^T y$$

Gain Adaptation

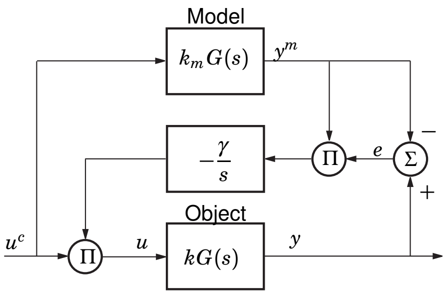


Figure: Gain adaptation, $u = \theta u_c$: How to change θ when k is unknown to get $\theta k = k_m$?

Gain Adaptation

Consider the gain adaptation problem of Fig. 1

$$u = \theta u^c$$

where the controlled system would be like the desired model if the gain parameter was

$$\theta = \theta^* = \frac{k_m}{k}$$

The output error is

$$e = y - y^m = G(s)(k\theta - k_m)u^c$$

with u^c as command signal, y^m the reference model output, y system output, θ the gain parameter.

$$\begin{aligned} \frac{d\theta}{dt} &= -\gamma y^m e \qquad \text{MIT} \\ \frac{d\theta}{dt} &= -\gamma u^c e \qquad \text{SPR} \end{aligned}$$

Lyapunov Stability

Assume that the transfer function $G(s)$ has a state-space realization

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx, \qquad Y(s) = G(s)U(s) \end{aligned}$$

and

$$\begin{aligned} \dot{x}_m &= Ax_m + B(k_m u^c) \\ y &= Cx_m, \qquad Y^m(s) = G(s)k_m U^c(s) \end{aligned}$$

The error model

$$\begin{aligned} x_e &= x - x_m \\ e &= y - y_m, \qquad E(s) = G(s)(k\theta - k_m)U^c(s) \end{aligned}$$

Lyapunov Stability (cont'd)

The error model

$$\begin{aligned} x_e &= x - x_m \\ e &= y - y_m, \qquad E(s) = G(s)(k\theta - k_m)U^c(s) \end{aligned}$$

with the error dynamics

$$\begin{aligned} \dot{x}_e &= Ax_e + B(k\theta - k_m)u^c = Ax_e + B\underbrace{ku^c}_{\phi}\tilde{\theta} \\ e &= Cx_e \end{aligned}$$

Introduce the Lyapunov function candidate

$$V(x_e, \tilde{\theta}) = \frac{1}{2}x_e^T P x_e + \frac{\mu}{2}\tilde{\theta}^T\tilde{\theta}, \quad P = P^T > 0, \, \mu > 0$$

Lyapunov Stability (cont'd)

Lyapunov function candidate

$$V(x_e, \tilde{\theta}) = \frac{1}{2}x_e^T P x_e + \frac{\mu}{2}\tilde{\theta}^T\tilde{\theta}, \quad P = P^T > 0, \, \mu > 0$$

with the derivative

$$\begin{aligned} \frac{dV(x_e, \tilde{\theta})}{dt} &= \frac{1}{2}x_e^T(PA + A^T P)x_e + x_e^T PB(k\theta - k_m)u^c + \mu\tilde{\theta}^T\frac{d\tilde{\theta}}{dt} \\ &= \frac{1}{2}x_e^T(PA + A^T P)x_e + \tilde{\theta}^T(B^T Pxku^c + \mu\frac{d\tilde{\theta}}{dt}) \end{aligned}$$

Lyapunov Stability (cont'd)

Under the conditions of the Kalman-Yakubovich-Popov (KYP) Lemma, we have for an SPR transfer function $G(s)$

$$\begin{aligned} PA + A^T P &= -Q, \qquad Q = Q^T > 0, \quad P = P^T > 0 \\ C &= B^T P \end{aligned}$$

then the adaptation law

$$\frac{d\widehat{\theta}}{dt} = -\gamma \underbrace{B^T P x_e}_{e} \underbrace{k u^c}_{\phi} = -\gamma \phi e, \qquad \gamma = \mu k$$

Lyapunov Stability (cont'd)

will render the Lyapunov function negative definite with respect to x_e, that is

$$\begin{aligned} \frac{dV(x_e, \widetilde{\theta})}{dt} &= \frac{1}{2} x_e^T (PA + A^T P) x_e \\ &= -\frac{1}{2} x_e^T Q x_e < 0, \qquad \|x_e\| \neq 0 \\ \frac{d\widetilde{\theta}}{dt} &= \frac{d\widehat{\theta}}{dt} = -\gamma \phi e \end{aligned}$$

Passivity Analysis

Passivity relationships require that

$$V(x(0)) + \int_0^t u^T(s) y(s) ds \geq V(x(t))$$

where $V(x(t))$ is a storage function.

An interpretation of the inequality (4) is a signal energy balance

Stored Energy $\leq$ Original Stored Energy $+$ Supplied Energy

Passivity Analysis (cont'd)

For the upper block with input $\widetilde{\theta}$, output e and storage function

$$V_x(x_e) = \frac{1}{2} x_e P x_e$$

we have (time arguments partly omitted)

$$\begin{aligned} u^T y - \frac{\partial V_x}{\partial x} \frac{d x_e}{dt} &= u^T y - x_e^T P (A x_e + B u) \\ &= u^T C x_e - x_e^T P (A x_e + B u) \\ &= -\frac{1}{2} x_e^T (PA + A^T P) x_e + u^T (C - B^T P) x_e \\ &= \frac{1}{2} x_e^T Q x_e \end{aligned}$$

Passivity Analysis (cont'd)

...so that

$$\begin{aligned} < u | e > &= \int_0^t u(s) y(s) ds = \frac{1}{2} \int_0^t x_e^T(s) Q x_e(s) ds + \int_0^t \frac{\partial V_x}{\partial x} \frac{d x_e}{dt} dt \\ &= \frac{1}{2} \int_0^t x_e^T(s) Q x_e(s) ds + V_x(x_e(t)) - V(x_e(0)) \\ &> V_x(x_e(t)) - V_x(x_e(0)) \end{aligned}$$

which satisfies the strict passivity conditions for a strictly positive real transfer function $G(s)$.

Passivity Analysis (cont'd)

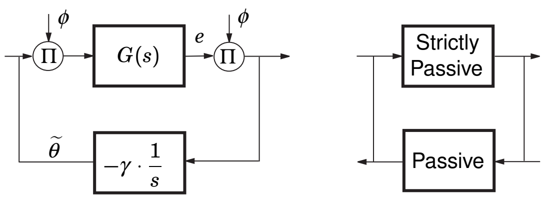


Figure: Passivity analysis of gain adaptation

Passivity Analysis (cont'd)

Similarly, for $\phi = u_c$

$$< \widetilde{\theta}\phi | e\phi > = \frac{1}{2} \int_0^t x_e^T(s) Q x_e(s) ds + V_x(x_e(t)) - V_x(x_e(0))$$

Furthermore, for the adaptation block with input $e\phi$ and output $-\widetilde{\theta}$ we have

$$< e\phi | -\widetilde{\theta} > = \int_0^t e(s)\phi(s)(-\widetilde{\theta}(s)) ds$$

Integration by parts gives

$$\begin{aligned} < e\phi | -\widetilde{\theta} > &= \int_0^t e(s)\phi(s)(-\widetilde{\theta}(s)) ds \\ &= \frac{1}{\gamma} \int_0^t \widetilde{\theta}^T(s) \widetilde{\theta}(s) ds + \frac{1}{\gamma} \int_0^t \widetilde{\theta}^T(s) \frac{d\widetilde{\theta}(s)}{ds} ds \end{aligned}$$

Passivity Analysis (cont'd)

Introduce the storage function

$$V_\theta(\widetilde{\theta}) = \frac{1}{2\gamma} \widetilde{\theta}^T \widetilde{\theta}$$

Passivity analysis verifies that

$$\begin{aligned} < e\phi | -\widetilde{\theta} > &= \int_0^t e(s)\phi(s)(-\widetilde{\theta}(s)) ds \\ &= \frac{1}{\gamma} \int_0^t \widetilde{\theta}^T(s) \widetilde{\theta}(s) ds + V_\theta(\widetilde{\theta}(t)) - V_\theta(\widetilde{\theta}(0)) \\ &\geq V_\theta(\widetilde{\theta}(t)) - V_\theta(\widetilde{\theta}(0)), \quad \gamma > 0 \end{aligned}$$

Passivity Analysis (cont'd)

A storage function for the passive feedback-interconnected system is

$$V(\xi) = V_x(x_e) + V_\theta(\widetilde{\theta}), \qquad \xi = \begin{bmatrix} x_e \\ \widetilde{\theta} \end{bmatrix}$$

with state vector ξ.

Passivity Analysis (cont'd)

A modified PI-type adaptation law may be suggested so that

$$\widetilde{\theta} = -\gamma_1 \phi(t) e(t) - \gamma_2 \int_0^t \phi(s) e(s) ds$$

For passivity analysis, we introduce the shorter notation

$$\widetilde{\theta} = \widetilde{\theta}_1 + \widetilde{\theta}_2, \text{ where } \widetilde{\theta}_1 = -\gamma_1 \phi e, \quad \widetilde{\theta}_2 = -\gamma_2 \int_0^t \phi(s) e(s) ds$$

Passivity Analysis (cont'd)

The input-output energy is

$$\begin{aligned} < e\phi | -\widetilde{\theta} > &= \int_0^t e(s)\phi(s)(-\widetilde{\theta}(s))ds \\ &= \int_0^t e(s)\phi(s)(-\widetilde{\theta}_1(s) - \widetilde{\theta}_2(s))ds \\ &= \frac{1}{\gamma_1}\int_0^t \widetilde{\theta}_1^2(s)ds + \int_0^t e(s)\phi(s)\widetilde{\theta}_2 ds \\ &= \frac{1}{\gamma_1}\int_0^t \widetilde{\theta}_1^2(s)ds + \frac{1}{\gamma_2}\int_0^t \widetilde{\theta}_2^2 ds + \frac{1}{2\gamma_2}\widetilde{\theta}_2^2(t) - \frac{1}{2\gamma_2}\widetilde{\theta}_2^2(0) \\ &> \frac{1}{2\gamma_2}\widetilde{\theta}_2^2(t) - \frac{1}{2\gamma_2}\widetilde{\theta}_2^2(0) \end{aligned}$$

Stability of MRAC

Assume that the control object can be described by the state equation

$$\begin{aligned} \dot{x} &= Ax + Bu = \\ &= \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ & I_{(n-1)\times(n-1)} & & 0_{(n-1)\times 1} \end{bmatrix} x \\ &+ \begin{bmatrix} 1 \\ 0_{(n-1)\times 1} \end{bmatrix} u \end{aligned}$$

and that

$$u = -\theta^T x$$

Stability of MRAC (cont'd)

In the case of a known A it is possible to choose a suitable θ by means of model matching so that

$$A - B\theta^T = A_m$$

for some dynamics matrix A_m representing the prescribed system behavior.

This gives the closed-loop system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ &= \begin{bmatrix} -a_1 - \theta_1 & -a_2 - \theta_2 & \cdots & -a_n - \theta_n \\ & I_{(n-1)\times(n-1)} & & 0_{(n-1)\times 1} \end{bmatrix} x = A_m x \end{aligned}$$

Stability of MRAC (cont'd)

Replace by the adaptive control law

$$\begin{aligned} \dot{\widetilde{\theta}} &= S^{-1} x B^T P x, \quad S = S^T > 0 \\ u &= -\widetilde{\theta}^T x \end{aligned}$$

where P solves the Lyapunov equation

$$PA_m + A_m^T P = -Q$$

The system behavior under adaptive feedback control is

$$\dot{x} = (A - B\theta^T)x - Bx^T\widetilde{\theta} = A_m x - Bx^T\widetilde{\theta}$$

Stability of MRAC (cont'd)

Lyapunov function candidate

$$V(x, \widetilde{\theta}) = \frac{1}{2}x^T P x + \frac{1}{2}\widetilde{\theta}^T S \widetilde{\theta}, \quad S = S^T > 0$$

with the derivative

$$\begin{aligned} \frac{dV}{dt} &= \frac{1}{2}\dot{x}^T P x + \frac{1}{2}x^T P \dot{x} + \frac{1}{2}\dot{\widetilde{\theta}}^T S \widetilde{\theta} + \frac{1}{2}\widetilde{\theta}^T S \dot{\widetilde{\theta}} \\ &= \frac{1}{2}x^T(A_m^T P + PA_m)x - x^T PBx^T\widetilde{\theta} + \widetilde{\theta}^T S \dot{\widetilde{\theta}} \end{aligned}$$

If θ is constant then $\dot{\widetilde{\theta}} = \dot{\widetilde{\theta}}$ and

$$\frac{dV}{dt} = -\frac{1}{2}x^T Q x < 0, \quad \|x\| \neq 0$$

Stability of MRAC (cont'd)

Consider adaptive stabilization of the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix} u$$

so that it behaves like the model

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -3 & -3 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A_m x$$

Stability of MRAC (cont'd)

Application of control algorithm for $Q = S = I_{3\times 3}$ and P solving the Lyapunov equation $PA_m + A_m P = -Q$ gives

$$P = \begin{bmatrix} 0.4375 & 0.8125 & 0.5000 \\ 0.8125 & 3.2500 & 1.9375 \\ 0.5000 & 1.9375 & 2.3125 \end{bmatrix} > 0$$

A simulation of this adaptive algorithm for $a_1 = a_2 = a_3 = -1$ and $b_1 = 1$ is shown in Fig. 3 in which typical transients of control and adaptation are exhibited.

Stability of MRAC (cont'd)

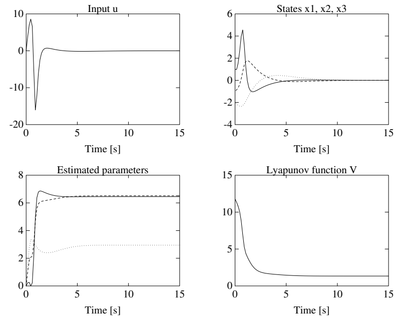
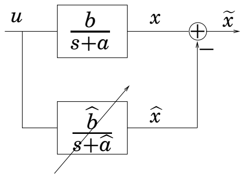


Figure: Example of Model reference adaptive control

References

Adaptive Noise Cancellation by Lyapunov Design



$$\dot{x} + ax = bu$$
$$\dot{\widehat{x}} + \widehat{a}\widehat{x} = \widehat{b}u$$

Introduce $\widetilde{x} = x - \widehat{x}$, $\widetilde{a} = a - \widehat{a}$, $\widetilde{b} = b - \widehat{b}$.

Want to design adaptation law so that $\widetilde{x} \to 0$

Let us try the Lyapunov function

$$V = \frac{1}{2}(\widetilde{x}^2 + \gamma_a \widetilde{a}^2 + \gamma_b \widetilde{b}^2)$$
$$\dot{V} = \widetilde{x}\dot{\widetilde{x}} + \gamma_a \widetilde{a}\dot{\widetilde{a}} + \gamma_b \widetilde{b}\dot{\widetilde{b}} =$$
$$= \widetilde{x}(-a\widetilde{x} - \widetilde{a}\widehat{x} + \widetilde{b}u) + \gamma_a \widetilde{a}\dot{\widetilde{a}} + \gamma_b \widetilde{b}\dot{\widetilde{b}} = -a\widetilde{x}^2$$

where the last equality follows if we choose

$$\dot{\widetilde{a}} = -\dot{\widehat{a}} = \frac{1}{\gamma_a}\widetilde{x}\widehat{x} \qquad \dot{\widetilde{b}} = -\dot{\widehat{b}} = -\frac{1}{\gamma_b}\widetilde{x}u$$

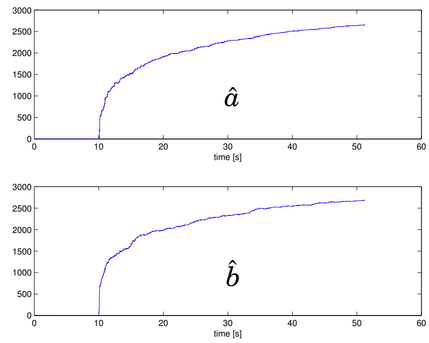
Invariant set: $\widetilde{x} = 0$.

This proves that $\widetilde{x} \to 0$.

(The parameters $\widetilde{a}$ and $\widetilde{b}$ do not necessarily converge: $u \equiv 0$.)

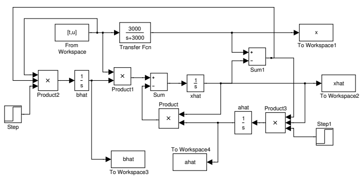
Demonstration if time permits

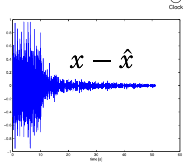
Results



Estimation of parameters starts at t=10 s.

Results





Estimation of parameters starts at t=10 s.