FRTN15 Predictive Control—Exercise Session 6

Solutions to FRTN15 Predictive Control—Exercise 6

1. The error between iteration is obtained as:

$$\begin{aligned} e_{k+1}(t) &= r(t) - y_{k+1}(t) = r(t) - G(q)u_{k+1}(t) \\ &= r(t) - G(q)u_k(t) - G(q)L(q)e_k(t) = (1 - G(q)L(q))e_k(t) \end{aligned}$$

The condition for the error not to grow is:

$$|1 - G(e^{i\omega})L(e^{i\omega})| < 1, \quad \forall \omega \in [-\pi,\, \pi]$$

From the Nyquist-plot results that this is not the case for $L(q) = 1$.

2.

a. A transfer function $G(s)$ is said to be *Positive Real* (PR) if:

$$\text{Re } G(s) \geq 0 \text{ for Re } s \geq 0$$

The transfer function is said to be *Strictly Positive Real* (SPR) if $G(s - \epsilon)$ is positive real for some $\epsilon > 0$. In terms of the Nyquist diagram, an SPR system must have a Nyquist curve which lies strictly in the right half-plane. For the case with

$$G(s) = \frac{1}{(s+1)}$$

we note that Re $G(s - \epsilon) \geq 0$ for Re $s \geq 0$ for any $\epsilon \leq 1$, which proves that the system is SPR.

To show that the storage function (Lyapunov function) given by:

$$V(x) = \frac{1}{2}x^T x$$

fullfills the passivity property, we begin by finding the time derivative:

$$\begin{aligned} \dot{V}(x(t)) &= \frac{1}{2}(\dot{x}^T x + x^T \dot{x}) \\ &= \frac{1}{2}((-x+u)^T x + x^T(-x+u)) \\ &= -x^T x + x^T u \end{aligned}$$

Since $y(t) = x(t)$ we may write:

$$\dot{V}(x(t)) = -x^T x + y^T u$$

At a specific time T the storage function is given by:

$$\begin{aligned} V(x(T)) &= V(x(0)) + \int_0^T \dot{V}(x(t))dt \\ &= V(x(0)) + \int_0^T (-x^T x + y^T u)dt \end{aligned}$$

Thus we obtain the passivity property:

$$V(x(T)) = V(x(0)) + \int_0^T y^T u dt - \int_0^T x^T x dt$$

as required. The term $V(x(0))$ represents the initial stored energy at time $t = 0$. The term $\int_0^T y^T u dt$ represents the energy supplied to the system between time $t = 0$ and $t = T$. To see this more clearly, it may be helpful to think of an electrical system where the input u is a voltage and the output y is a current. Power is given by $P = VI$ and the energy supplied to the system is thus the integral of this. The last term $\int_0^T x^T x$ represents the energy dissipated by the system. Returning to the electrical system example, we know that dissipated power is given by I^2/R, so the dissipated energy is therefore the integral of this.

In summary, a system is passive if the change in stored energy is equal to the energy supplied minus the energy dissipated.

b. By looking at the Nyquist plot of the system we see that Re $G(s)$ takes negative values for certain ω when Re $s = 0$. The system is therefore not positive real.

3.

a. When using the backstepping method, we aim to stabilize one first-order subsystem at a time, through the use of 'virtual controls'. Here, we begin by looking at the x_1 subsystem:

$$\dot{x_1} = -x_2 + \theta x_1^2$$

and regarding the state x_2 as a virtual control. In effect, we regard the system as if x_2 were the control signal and design a control law which stabilizes the system. In the problem, the state x_2 is the integral of the actual control signal. This means that in order to find the actual control signal, we must 'step back' through the integrator. Hence the name 'backstepping'.

We begin by defining error states, here denoted by z. The first component is given by $z_1 = x_1$. We have:

$$\frac{dz_1}{dt} = -x_2 + \theta z_1^2$$

In order to introduce stable dynamics we may add and subtract the term z_1 to the right hand side:

$$\frac{dz_1}{dt} = -z_1 + z_1 - x_2 + \theta z_1^2$$

We see that if we choose the virtual control x_2 to be:

$$x_2 = z_1 + \theta z_1^2$$

then the error dynamics become:

$$\frac{dz_1}{dt} = -z_1$$

We may now introduce the second error coordinate:

$$z_2 = z_1 + \theta z_1^2 - x_2$$

i.e., the difference between the virtual control and its desired value. Thus we obtain:

$$\frac{dz_1}{dt} = -z_1 + z_2$$

We now find the dynamics of the second error coordinate:

$$\frac{dz_2}{dt} = \frac{dz_1}{dt} - \frac{dx_2}{dt} + 2\theta z_1 \frac{dz_1}{dt}$$

As before, we may add and subtract the term z_2 to the right hand side, and replace $\frac{dx_2}{dt}$ with the control signal u

$$\frac{dz_2}{dt} = -z_2 + z_2 + \frac{dz_1}{dt} - u + 2\theta z_1 \frac{dz_1}{dt}$$

Therefore if we choose:

$$u = z_2 + \frac{dz_1}{dt} + 2\theta z_1 \frac{dz_1}{dt}$$

we obtain:

$$\frac{dz_2}{dt} = -z_2$$

Thus the closed loop system of error coordinates becomes:

$$\begin{bmatrix} \dot{z_1} \\ \dot{z_2} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

which is a stable system.

b. In this case the parameter θ is no longer known, and may be time–varying. A parameter update law will therefore be required, along with a control law.

As in the standard backstepping problem, we define the first error coordinate by $z_1 = x_1$. We will also define the parameter estimation error:

$$\tilde{\theta} = \theta - \hat{\theta}$$

Adding and subtracting the term z_1 as in the first problem, and rewriting θ in terms of $\tilde{\theta}$ and $\hat{\theta}$, we obtain:

$$\frac{dz_1}{dt} = -z_1 + z_1 - x_2 + \hat{\theta} z_1^2 + \tilde{\theta} z_1^2$$

Using x_2 as a virtual control, with:

$$x_2 = z_1 + \hat{\theta} z_1^2$$

we can define the next error coordinate z_2:

$$z_2 = x_2 - z_1 - \hat{\theta} z_1^2$$

We therefore obtain:

$$\frac{dz_1}{dt} = -z_1 - z_2 + \tilde{\theta} z_1^2$$

Notice that the term containing $\tilde{\theta}$ could not be included in the virtual control since it is not known. We see here that we require that $\tilde{\theta}$ tends to zero in order to obtain stable error dynamics.

Taking the derivative of z_2:

$$\frac{dz_2}{dt} = -z_2 + z_2 + \frac{dx_2}{dt} - \frac{dz_1}{dt} - 2z_1\hat{\theta}\frac{dz_1}{dt} - z_1^2\frac{d\hat{\theta}}{dt}$$

Replacing $\frac{dx_2}{dt}$ with the control signal u and substituting for $\frac{dz_1}{dt}$:

$$\frac{dz_2}{dt} = -z_2 + z_2 + u - (-z_1 - z_2 + \tilde{\theta} z_1^2) - 2z_1\hat{\theta}(-z_1 - z_2 + \tilde{\theta} z_1^2) - z_1^2\frac{d\hat{\theta}}{dt}$$

Here we cannot simply choose a control signal to remove all the unwanted terms since the signal $\tilde{\theta}$ is unknown. We must therefore use Lyapunov theory to derive a stabilizing controller and a stable parameter update law. Consider the Lyapunov function:

$$V(z, \tilde{\theta}) = \frac{1}{2}(z_1^2 + z_2^2 + \tilde{\theta}^2)$$

This may be regarded as a Lyapunov function for the system consisting of the error coordinates z and the paramter error $\tilde{\theta}$. If we can find a controller and a parameter update law which make the time derivative of this Lyapunov function negative, then we have solved the design problem.

The time derivative is given by:

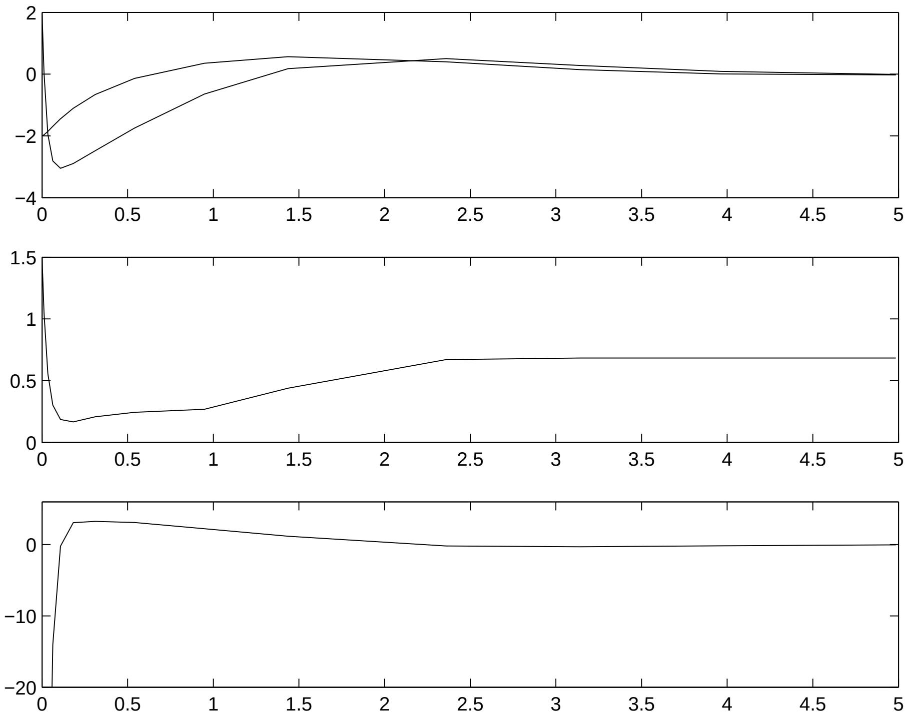
$$\frac{dV(z, \tilde{\theta})}{dt} = z_1\frac{dz_1}{dt} + z_2\frac{dz_2}{dt} + \tilde{\theta}\frac{d\tilde{\theta}}{dt}$$

Substituting for the state derivatives, we obtain:

$$\begin{aligned}\frac{dV(z, \tilde{\theta})}{dt} =& z_1(-z_1 - z_2 + \tilde{\theta} z_1^2) + z_2(-z_2 + z_2 + u - (-z_1 - z_2 + \tilde{\theta} z_1^2) \\ & - 2z_1\hat{\theta}(-z_1 - z_2 + \tilde{\theta} z_1^2) - z_1^2\frac{d\hat{\theta}}{dt}) - \tilde{\theta}\frac{d\hat{\theta}}{dt}\end{aligned}$$

Expanding:

$$\begin{aligned}\frac{dV(z, \tilde{\theta})}{dt} =& - z_1^2 - z_2 z_1 + \tilde{\theta} z_1^3 + z_2(-z_2 + z_2 + u + z_1 + z_2 - \tilde{\theta} z_1^2 + 2z_1^2\hat{\theta} \\ & - 2z_2 z_1\hat{\theta} + 2z_1^3\hat{\theta}\tilde{\theta} - z_1^2\frac{d\hat{\theta}}{dt}) - \tilde{\theta}\frac{d\hat{\theta}}{dt}\end{aligned}$$



Figur 1 Simulation results showing states (top), parameter value (middle) and control signal (bottom)

Rewriting:

$$\frac{dV(z,\tilde{\theta})}{dt} = -z_1^2 - z_2^2 + \tilde{\theta}(z_1^3 - z_2 z_1^2 - 2z_1^3 z_2 \hat{\theta} - \frac{d\hat{\theta}}{dt}) + z_2(2z_2 + u + 2z_1^2\hat{\theta} + 2z_2 z_1 \hat{\theta} - z_1^2 \frac{d\hat{\theta}}{dt})$$

If we now choose u and $\frac{d\hat{\theta}}{dt}$ such that the brackets containing them vanish, the Lyapunov function time derivative will be negative as desired. Thus we have:

$$\frac{d\hat{\theta}}{dt} = z_1^3 - z_1^2 z_2(1 + 2z_1\hat{\theta})$$

$$u = -2z_2 - 2z_1\hat{\theta}(z_1 + z_2) + z_1^2 \frac{d\hat{\theta}}{dt}$$

Figure shows the results of simulating the system. We see that both the states and the parameter estimate converge, as desired.