LUNDS TEKNISKA
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Institutionen för
REGLERTEKNIK

FRTN15 Predictive Control

Final Exam October 19, 2010, 14-19

General Instructions

This is an open book exam. You may use any book you want. However, no previous exam sheets or solution manuals are allowed. The exam consists of 6 problems to be solved. Your solutions and answers to the problems should be well motivated. The credit for each problem is indicated in the problem. The total number of credits is 25 points. Preliminary grade limits are:

Grade 3:	12 points
Grade 4:	17 points
Grade 5:	22 points

Results

The results of the exam will be posted at the latest October 28 on the notice board on the first floor of the M-building and they will also be available on the course home page.

Do you accept publication of your grading result on our local web page? (Godkänner du publicering av resultatet på vår lokala hemsida?)

1.

a. What is the basic principle of Model Reference Control? What design choices have to be made? (2 p)

b. Consider the process $G(z)$ given by:

$$G(z) = \frac{b_0 z}{z^2 + a_1 z + a_2}$$

Design a Model Reference Controller for the process $G(z)$, which includes integral action, so that the closed-loop system is given by the reference model:

$$G_m(z) = \frac{b_{m0} z + b_{m1}}{z^2 + a_{m1} z + a_{m2}}$$

(2 p)

c. What can be done if the structure of the process $G(z)$ is known, but not its parameters? (1 p)

Solution

a. For a process of the form:

$$A(z) \cdot y_k = B(z) \cdot u_k + C(z) \cdot v_k$$

a controller

$$R(z) \cdot u_k = T(z) \cdot u_k^c - S(z) \cdot y_k$$

should be designed, so that the closed-loop response

$$Y(z) = \frac{B(z) \cdot T(z)}{A(z) \cdot R(z) + B(z) \cdot S(z)} \cdot U^c(z)$$

matches the desired reference model

$$Y_m(z) = \frac{B_m(z)}{A_m(z)} \cdot U^c(z).$$

The design choices that have to be made are:

- choosing a reference model $G_m(z)$.
- deciding whether or not to cancel process zeros
- the degree of the observer polynomial

b. Since the zero is stable, it can be canceled:

$$B^+ = z \text{ and } B^- = b_0.$$

Because of the integral action to be included into the controller, we have to include a model of the load-disturbance into the controller:

$$R = (z-1) \cdot R_1 \cdot B^+ = (z-1) \cdot R_1 \cdot z.$$

The closed-loop polynomial is:

$$A_c = A_0 \cdot A_m \cdot B^+ = A_0 \cdot A_m \cdot z.$$

The B^- polynomial, here just a gain, is extracted for the numerator of the reference system:

$$B_m = B^- \cdot B_{1m}.$$

Therefore it is:

$$B_{1m} = \frac{b_{m0}}{b_0} z + \frac{b_{m1}}{b_0}.$$

With this, the Diophantine Equation to determine the polynomials $R(z)$ and $S(z)$ is:

$$A \cdot (z-1) \cdot R_1 + B^- \cdot S = A_0 \cdot A_m.$$

The degree conditions are as follows:

$$deg(R) = deg(z-1) + deg(z) + deg(R_1) = deg(R_1) + 2,$$

when choosing $deg(R_1) = 0$, then it follows that $deg(R) = 2$. Because of causality we have:

$$deg(S) = deg(R) = 2 \text{ and } deg(T) = deg(R) = 2.$$

The Diophantine Equation now is:

$$(z^2 + a_1 z + a_2)(z-1)r_0 + b_0(s_0 z^2 + s_1 z + s_2) = (z + a_0^1)(z^2 + a_{m1} z + a_{m2}).$$

Sorting by orders gives:

$$\begin{aligned}
r_0 &= 1 \\
r_0 a_1 - r_0 + b_0 s_0 &= a_{m1} + a_0^1 \\
r_0 a_2 - r_0 a_1 + b_0 s_1 &= a_{m2} + a_0^1 a_{m1} \\
b_0 s_2 - r_0 a_2 &= a_0^1 a_{m2},
\end{aligned}$$

which leads to the solution:

$$\begin{aligned}
r_0 &= 1 \\
s_0 &= \frac{1}{b_0}(a_{m1} + a_0^1 - a_1 + 1) \\
s_1 &= \frac{1}{b_0}(a_{m2} + a_0^1 a_{m1} - a_2 + a_1) \\
s_2 &= \frac{1}{b_0}(a_0^1 a_{m2} + a_2).
\end{aligned}$$

The polynomials R and S are then:

$$\begin{aligned}
S(z) &= s_0 z^2 + s_1 z + s_2 \\
R(z) &= (z-1)z
\end{aligned}$$

The polynomials T follows from model following:

$$\frac{BT}{AR + BS} = \frac{BT}{A_c} = \frac{B^- T}{A_0 A_m} = \frac{B_m}{A_m},$$

so that

$$T(z) = \frac{B_m}{B^-} A_0 = B_{1m} A_0 = \frac{1}{b_0}(b_{m0}z + b_{m1})(z + a_0^1).$$

The controller is then given by:

$$R(z)u_k = T_z u_k^c - S(z)y_k.$$

c. The parameters of the process can be estimated using (recursive) least squares estimation.

2. Consider the a process described by:

$$A(z^{-1})y_k = B(z^{-1})u_k + e_k$$

where e_k is Gaussian white noise with variance σ^2 and

$$\begin{aligned} A(z^{-1}) &= 1 + a_1 z^{-1} + a_2 z^{-2} \\ B(z^{-1}) &= b_0 z^{-1} + b_1 z^{-2} \end{aligned}$$

a. Determine the regressor and parameter vector for least-squares identification of the unknown parameters b_0, b_1, a_1 and a_2. Also, derive the normal equations to calculate these parameters. (2 p)

b. How does the algorithm from a) need to be changed to provide new parameter estimates continuously in a real-time setting?
What problem can arise when the parameters are time-variant? How can the algorithm be modified to improve the performance in case of time-varying parameters? (2 p)

Solution

a. The process can be written as:

$$y_k = -a_1 y_{k-1} - a_2 y_{k-2} + b_0 u_{k-1} + b_1 u_{k-2},$$

which can be expressed as:

$$y_k = \phi_k^T \cdot \theta$$

with

$$\phi_k = \begin{pmatrix} -y_{k-1} & -y_{k-2} & u_{k-1} & u_{k-2} \end{pmatrix}^T$$

and

$$\theta = \begin{pmatrix} a_1 & a_2 & b_0 & b_1 \end{pmatrix}^T,$$

where θ is the parameter vector and ϕ_k the regressor.

Now assume that N data points have been measured. Then the data can be collected as follows:

$$Y_N = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_N \end{pmatrix}$$

$$\Phi_N = \begin{pmatrix} \phi_2 \\ \phi_3 \\ \vdots \\ \phi_N \end{pmatrix} = \begin{pmatrix} -y_1 & -y_0 & u_1 & u_0 \\ -y_2 & -y_1 & u_2 & u_1 \\ \vdots & & & \\ -y_{N-1} & -y_{N-2} & u_{N-1} & u_{N-2} \end{pmatrix}$$

With this, the process can be described by:

$$Y_N = \Phi_N \cdot \theta.$$

Multiplying both sides of the last equation from the left with Φ_N^T leads to the normal equation that can be used to calculate the unknown parameters:

$$\begin{aligned} \Phi_N^T \cdot Y_N &= \left(\Phi_N^T \Phi_N\right) \theta \\ \Rightarrow \hat{\theta} &= \left(\Phi_N^T \Phi_N\right)^{-1} \cdot \Phi_N^T \cdot Y_N. \end{aligned}$$

Also, the Least-Squares Estimation aims to minimize teh sum of squared errors between the model output and the observations:

$$V(\hat{\theta}) = \frac{1}{2}\epsilon^T \epsilon = \frac{1}{2}(Y_N - \Phi_N \hat{\theta})^T (Y_N - \Phi_N \hat{\theta}).$$

The minimum can be found for the estimate

$$\hat{\theta} = \left(\Phi_N^T \Phi_N\right)^{-1} \cdot \Phi_N^T \cdot Y_N.$$

This can be seen by taking the first derivative of $V(\hat{\theta})$ and setting it to zero (this is the normal equation):

$$\frac{\partial V(\hat{\theta})}{\partial \hat{\theta}} = -Y_N^T \Phi_N + \hat{\theta} \left(\Phi_N^T \Phi_N\right) = 0$$

which solved for $\hat{\theta}$ gives the optimal estimate:

$$\hat{\theta} = \left(\Phi_N^T \Phi_N\right)^{-1} \cdot \Phi_N^T \cdot Y_N.$$

b. In order to continuously provide updated parameters, the least-squares algorithm can be formulated in a recursive way (=Recursive Least-Squares). When estimating the unknown parameters at a specific time point, older information that is no longer relevant (e.g., before the parameters changed) could influence the parameter estimation and thus lead to poor estimation performance in case of time-varying parameters. The performance can be improved by neglecting older information. This can be done by including the "forgetting factor" λ into the Recursive Least-Squares algorithm, which provides an exponentially decreasing weight on old data.
Another possibility to improve the performance for time-varying parameters is to interpret the least-squares estimation as a Kalman filter by assuming a time-varying mathematical model for the parameters (see for example p. 59 in the textbook). The parameters now do not converge exponentially anymore, but in a linear way.
If it is known that the parameters are constant over a long period of time and change abruptly only occasionally, it is more suitable to reset the covariance matrix of the estimator when the change of parameters occurs (=covariance resetting) or periodically.

3. Consider the 1-step-ahead prediction of a process given as:

$$y_{k+1} = -0.5y_k + 2.5y_{k-1} + u_k + 0.8u_{k-1} + w_{k+1} + 0.1w_k$$

where w_k is white noise with $E\{w_k w_j^T\} = \sigma_w^2 \cdot \delta_{kj}$.

a. Calculate the 1-step-ahead predictor, its prediction error covariance and the 1-step-ahead minimum-variance controller for this process. (3 p)

b. Show that the controller calculated in a) minimizes the variance of the 1-step-ahead output. (2 p)

Solution

a. The process can be written as:

$$Y(z)(1+0.5z^{-1}-2.5z^{-2}) = z^{-1}(1+0.8z^{-1})U(z) + (1+0.1z^{-1})W(z)$$

$$\Rightarrow Y(Z) = \underbrace{\frac{z^{-1}\cdot(1+0.8z^{-1})}{1+0.5z^{-1}-2.5z^{-2}}}_{=\frac{B(z^{-1})}{A(z^{-1})}} U(z) + \underbrace{\frac{1+0.1z^{-1}}{1+0.5z^{-1}-2.5z^{-2}}}_{=\frac{C(z^{-1})}{A(z^{-1})}} W(z)$$

The Diophantine Equation to determine the polynomials $F(z^{-1})$ and $G(z^{-1})$ needed for the 1-step-ahead predictor and the minimum-variance controller is:

$$C(z^{-1}) = A(z^{-1})F(z^{-1}) + z^{-1}G(z^{-1}).$$

The orders of these polynomials have to be $deg(F) = 0$ and $deg(G) = 1$. Hence, the Diophantine Equation is:

$$(1+0.1z^{-1}) = (1+0.5z^{-1}-2.5z^{-2})f_0 + z^{-1}(g_0 + g_1 z^{-1}).$$

Sorting for degrees of z^{-1} gives:

$$\begin{aligned} 0 &= g_1 - 2.5f_0 \\ 0.1 &= g_0 + 0.5f_0 \\ 1 &= f_0, \end{aligned}$$

which gives

$$F(z^{-1}) = 1 \text{ and } G(z^{-1}) = -0.4 + 2.5z^{-1}$$

Hence, the 1-step-ahead predictor is:

$$\hat{y}_{k+1|k} = \frac{G(z^{-1})}{C(z^{-1})}y_k + \frac{B(z^{-1})F(z^{-1})}{C(z^{-1})}u_k = \frac{-0.4+2.5z^{-1}}{1+0.1z^{-1}}y_k + \frac{1+0.8z^{-1}}{1+0.1z^{-1}}.$$

The prediction error covariance for this predicted output is:

$$E\{(y_{k+1} - \hat{y}_{k+1})^2\}$$

$$= E\{(F(z^{-1})w_{k+1} + \frac{G(z^{-1})}{C(z^{-1})}y_k + \frac{B(z^{-1})F(z^{-1})}{C(z^{-1})}u_k - \frac{G(z^{-1})}{C(z^{-1})}y_k - \frac{B(z^{-1})F(z^{-1})}{C(z^{-1})}u_k)^2\}$$

$$= E\{1 \cdot w_{k+1}^2\} = \sigma_w^2$$

The minimum-variance controller is here:

$$U(z) - \frac{G(z^{-1})}{B(z^{-1})F(z^{-1})}Y(z) = -\frac{(-0.4 + 2.5z^{-1})}{1 + 0.8z^{-1}}Y(z)$$

b. The 1-step-ahead output of the process is as in a):

$$y_{k+1} = \frac{B(z^{-1})}{A(z^{-1})}u_k + \frac{C(z^{-1})}{A(z^{-1})}w_{k+1}.$$

Inserting the Diophantine Equation $C(z^{-1}) = A(z^{-1})F(z^{-1}) + z^{-1}G(z^{-1})$ into the last equation and expressing the unknown noise as $w_k = \frac{A(z^{-1})}{C(z^{-1})}y_k - z^{-1}\frac{B(z^{-1})}{C(z^{-1})}u_k$ leads to:

$$y_{k+1} = F(z^{-1})w_{k+1} + \frac{G(z^{-1})}{C(z^{-1})}y_k + [\frac{B(z^{-1})}{A(z^{-1})} - z^{-1}\frac{B(z^{-1})}{C(z^{-1})}\frac{G(z^{-1})}{A(z^{-1})}]u_k.$$

The polynominal in the third sum on the right hand side of the last equation can be exchanged by $G(z^{-1}) = zC(z^{-1}) - zA(z^{-1})F(z^{-1})$. This leads to the 1-step-ahead output of the output where future noise and past input-/output values are decoupled:

$$y_{k+1} = F(z^{-1})w_{k+1} + \frac{G(z^{-1})}{C(z^{-1})}y_k + \frac{B(z^{-1})F(z^{-1})}{C(z^{-1})}u_k.$$

The variance of the 1-step-ahead output is:

$$\begin{aligned}
E\{y_{k+1|k}^2\} &= E\{(F(z^{-1})w_k + \frac{G(z^{-1})}{C(z^{-1})}y_k + \frac{B(z^{-1})F(z^{-1})}{C(z^{-1})}u_k)^2\} \\
&= E\{(F(z^{-1})w_k)^2\} + E\{(\frac{G(z^{-1})}{C(z^{-1})}y_k + \frac{B(z^{-1})F(z^{-1})}{C(z^{-1})}u_k)^2\} \\
&= \sigma_w^2 + E\{(\frac{0.4 + 2.5z^{-1}}{1 + 0.1z^{-1}}y_k + \frac{1 + 0.8z^{-1}}{1 + 0.1z^{-1}}u_k)\}.
\end{aligned}$$

This variance has its minimum if:

$$\begin{aligned}
\frac{-0.4 + 2.5z^{-1}}{1 + 0.1z^{-1}}y_k + \frac{1 + 0.8z^{-1}}{1 + 0.1z^{-1}}u_k &= 0 \\
\Leftrightarrow u_k &= -\frac{-0.4 + 2.5z^{-1}}{1 + 0.8z^{-1}}y_k,
\end{aligned}$$

which is the minimum-variance controller from a).

4. The following system is to be controlled using Model Predictive Control

$$\begin{aligned}
x_{k+1} &= x_k + u_k \\
y_k &= 2x_k
\end{aligned}$$

There are constraints on the output and the rate of change of the control signal, $\Delta u_k = u_k - u_{k-1}$ according to

$$\begin{aligned}
y^{min} &\leq y_k \leq y^{max} \\
\Delta u^{min} &\leq \Delta u_k \leq \Delta u^{max}
\end{aligned}$$

where $y^{min} = -5$, $y^{max} = 5$, $\Delta u^{min} = -2$, $\Delta u^{max} = 2$.

a. Determine what values of u_k that are feasible when $x_k = 1$, and $u_{k-1} = 2$. (2 p)

b. Assume a measurement $y_k = 10$ is received when $u_{k-1} = 0$. Show that there are no feasible control moves Δu_k. (1 p)

c. Suggest a change in how the controller is implemented to avoid the situation in **b**. (1 p)

d. Show how the model can be modified to achieve integral action through the use of a disturbance observer, i.e. by assuming a constant disturbance acting on the input. (1 p)

Solution

a. The output at cycle k+1 is

$$y_{k+1} = 2x_{k+1} = 2(x_k + u_k) = 2 + 2u_k$$

so the constraint on y_{k+1} becomes

$$-5 \leq 2 + 2u_k \leq 5$$

which can be rearranged to

$$-1.5 = \frac{-5-2}{2} \leq u_k \leq \frac{5-2}{2} = 1.5$$

Using $\Delta u_k = u_k - u_{k-1}$, the constraint on Δu_k can be rewritten as

$$0 = -2 + 2 \leq u_k \leq 2 + 2 = 4$$

This yields the effective constraints

$$0 \leq u_k \leq 1.5$$

b. When $y_k = 10$ we have $x_k = 5$. The constraint on y_{k+1} becomes

$$-5 \leq 10 + 2(0 + \Delta u_k) \leq 5$$

This imposes the following constraint on Δu_k

$$-7.5 = \frac{-5-10}{2} \leq \Delta u_k \leq \frac{5-10}{2} = -2.5$$

which is outside the constraints on Δu_k.

c. By introducing slack variables that penalize constraint violations (soft constraints) and removing the hard constraint on y_k, a feasible solution could still be found.

d. With a constant input disturbance d_k, the model takes the form

$$
\begin{aligned}
x_{k+1} &= x_k + u_k + d_k \\
y_k &= 2x_k
\end{aligned}
$$

By introducing an extended state vector $\begin{pmatrix} x_k & d_k \end{pmatrix}^T$ we can write this as

$$
\begin{aligned}
\begin{pmatrix} x_{k+1} \\ d_{k+1} \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_k \\ d_k \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_k \\
y_k &= \begin{pmatrix} 2 & 0 \end{pmatrix} \begin{pmatrix} x_k \\ d_k \end{pmatrix}
\end{aligned}
$$

Basing the observer and controller on this model will achieve integral action in the closed loop system.

5. Iterative Learning control for a system described by

$$y_k(t) = G(q)u_k(t)$$

can be implemented as

$$
\begin{aligned}
e_k(t) &= r_k(t) - y_k(t) \\
u_{k+1}(t) &= u_k(t) + L(q)e_k(t)
\end{aligned}
$$

a. Give an interpretation of the controller equations. What is the role of $L(q)$? (1 p)

b. For what type of control problems is ILC a suitable strategy? What types of disturbances can be handled? (1 p)

c. What condition on $L(q)$ must hold for the control error to converge? Give a graphical interpretation of this in terms of the Nyquist plot of $L(q)G(q)$. (1 p)

Solution

a. The control signal in the next iteration is given by that of the previous iteration modified by the control signal filtered by $L(q)$. The filter $L(q)$ determines how big the impact of the last run is on the control signal in the next iteration.

b. The control problem should be of a repetitive nature where the same task has to be repeated over and over. Examples of this include manufacturing using robots where the same part is constructed many times.

Any repeating disturbances can be handled since the control signal will adapt to them. More stochastic disturbances are harder to handle since the error sequence of the last iteration doesn't include any information about the disturbance.

c. The condition is

$$\sup_{wh \in [-\pi,\pi]} ||I - G(z)L(z)||_{|z=e^{i\omega h}} < 1$$

This can be expressed graphically as requiring that the Nyquist plot $L(z)G(z)$ is contained in unit circle centered at $z = 1$.

6. A stochastic problem is generated as

$$\begin{aligned} x_{k+1} &= 0.9x_k + v_k \\ y_k &= 0.1x_k + e_k \end{aligned}$$

with uncorrelated white-noise processes e_k and v_k. The covariances of these noise processes are $E\{v_k v_j^T\} = \sigma_v^2 \cdot \delta_{kj}$ and $E\{e_k e_j^T\} = \sigma_e^2 \cdot \delta_{kj}$. Furthermore, x_0 is normally distributed with zero mean and variance σ_0^2.

a. Determine the Kalman Filter for the process. (1 p)

b. Determine the estimation covariance P and the filter gain K in case of a stationary Kalman Filter.
Hint: You do not need to calculate the estimation covariance exactly. It is sufficient to give an equation that the estimation covariance can be calculated from including conditions for an admissible solution. (1 p)

c. What are the estimation covariance P and the filter gain K in steady-state when $\sigma_e^2 >> \sigma_v^2$ is true for the noise-processes? Also, interpret the result. (1 p)

Solution

a. The Kalman Filter is given in Table 7.1 in the Textbook:

$$\begin{aligned} \hat{x}_{k+1|k} &= 0.9\hat{x}_{k|k-1} + K_k(y_k - \hat{y}_{k|k-1}) \\ \hat{y}_{k|k-1} &= 0.2\hat{x}_{k|k-1} \\ K_k &= \frac{0.9P_{k|k-1}0.2}{\sigma_e^2 + 0.2P_{k|k-1}0.2} = \frac{0.18P_{k|k-1}}{\sigma_e^2 + 0.04P_{k|k-1}} \\ P_{k+1|k} &= 0.9P_{k|k-1}0.9 + \sigma_v^2 - 0.9P_{k|k-1}0.2R_k^{-1}0.2P_{k|k-1}0.9 \\ R_k &= \sigma_e^2 + 0.2P_{k|k-1}0.2 \\ \Rightarrow P_{k+1|k} &= 0.81P_{k|k-1} + \sigma_v^2 - \frac{0.0324P_{k|k-1}^2}{\sigma_e^2 + 0.04P_{k|k-1}} \end{aligned}$$

b. In case of stationarity, the estimation covariance is:

$$P = \lim_{k\to\infty} P_{k|k-1} = \lim_{k\to\infty} P_{k+1|k}.$$

Therefore it is:

$$P = 0.81P + \sigma_v^2 - \frac{0.0324P^2}{\sigma_e^2 + 0.04P}. \tag{1}$$

The estimation covariance is then the solution to the equation:

$$P^2 + (4.75Q - \sigma_v^2)P + 25Q\sigma_v^2.$$

The filter gain is:

$$K = \frac{0.18P}{\sigma_e^2 + 0.04P}.$$

c. To analyze the behavior of the Kalman filter for $\sigma_e^2 >> \sigma_v^2$, we consider the extreme case where $\sigma_e^2 \to \infty$. For this case, the equation (1) gets:

$$\begin{aligned} P &= 0.81P + \sigma_v^2 \\ P &= \frac{1}{1-0.81}\sigma_v^2 = 0.19\sigma_v^2. \end{aligned}$$

The filter gain for this case gets $K = 0$.
Therefore, the poles of the Kalman filter are at $det(A - KC) = (0.9 - 0 \cdot 0.2) = 0.9$. The state estimation is given with:

$$\begin{aligned} \hat{x}_{k+1|k} &= 0.9\hat{x}_{k|k-1} \\ \hat{y}_{k|k-1} &= 0.2\hat{x}_{k|k-1}. \end{aligned}$$

The state and output of the process estimated by the Kalman filter depends only on the states estimated at the previous time-point when $\sigma_e^2 >> \sigma_v^2$. The filter thus assumes there is significantly more measurement noise present as process noise, so that it is better to rely on the estimated states and not on the measured process output.