LUND INSTITUTE
OF TECHNOLOGY
Lund University

Department of
AUTOMATIC CONTROL

FRTN15 Predictive Control

Final Exam October 23, 2009, 08-13

General Instructions

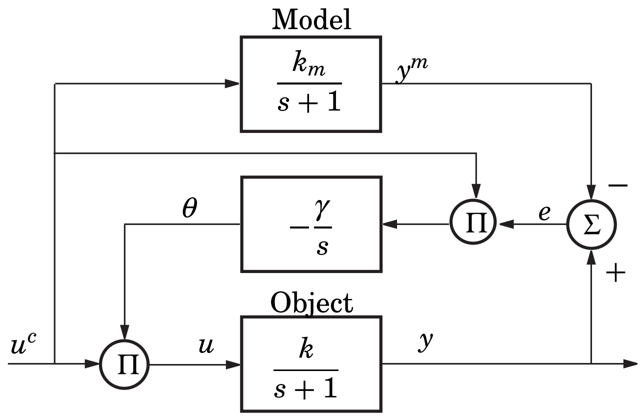
This is an open book exam. You may use any book you want. However, no previous exam sheets or solution manuals are allowed. The exam consists of 6 problems to be solved. Your solutions and answers to the problems should be well motivated. The credit for each problem is indicated in the problem. The total number of credits is 25 points. Preliminary grade limits are:

Grade 3:	12 points
Grade 4:	17 points
Grade 5:	22 points

Results

The results of the exam will be posted at the latest October 26 on the notice board on the first floor of the M-building and they will also be available on the course home page.

Do you accept publication of your grading result on our local web page? (Godkänner du publicering av resultatet på vår lokala hemsida?)

1. 

Solutions to Predictive Control exam, October 16, 2007

Consider the gain adaptation problem of Fig. 1 for $k > 0$

$$u = \theta u^c$$

Introduce the gain parameter

$$\theta = \frac{k_m}{k}$$

and the output error

$$e = y - y^m = G(s)k\theta u^c - k_m G(s)u^c, \qquad G(s) = \frac{1}{s+1}$$

with u^c as command signal, y^m the reference model output, y system output, θ the gain parameter.

$$\frac{d\theta}{dt} = -\gamma u^c e$$

Show that the gain adaptation is stable in the sense of Lyapunov for $\gamma > 0$. (2 p)

Solution

Assume that the transfer function $G(s)$ has a state-space realization

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx, \qquad Y(s) = G(s)U(s) \end{aligned}$$

and

$$\begin{aligned} \dot{x}_m &= Ax_m + B(k_m u^c) \\ y_m &= Cx_m, \qquad Y^m(s) = G(s)k_m U^c(s) \end{aligned}$$

The error model

$$\begin{aligned} x_e &= x - x_m \\ e &= y - y_m, \qquad E(s) = G(s)(k\theta - k_m)U^c(s) \end{aligned}$$

with the error dynamics

$$\begin{aligned}\dot{x}_e &= Ax_e + B(k\theta - k_m)u^c = Ax_e + B\underbrace{ku^c}_{\phi}\widetilde{\theta} \\ e &= Cx_e\end{aligned}$$

Introduce the Lyapunov function candidate

$$V(x_e, \widetilde{\theta}) = \frac{1}{2}x_e^T P x_e + \frac{\mu}{2}\widetilde{\theta}^T\widetilde{\theta}, \quad P = P^T > 0,\, \mu > 0$$

with the derivative

$$\begin{aligned}\frac{dV(x_e, \widetilde{\theta})}{dt} &= \frac{1}{2}x_e^T(PA + A^T P)x_e + x_e^T PB(k\theta - k_m)u^c + \mu\widetilde{\theta}^T\frac{d\widetilde{\theta}}{dt} \\ &= \frac{1}{2}x_e^T(PA + A^T P)x_e + \widetilde{\theta}^T(B^T Pxku^c + \mu\frac{d\widetilde{\theta}}{dt})\end{aligned}$$

Under the conditions of the Kalman-Yakubovich-Popov (KYP) Lemma, we have for an SPR transfer function $G(s)$

$$\begin{aligned}PA + A^T P &= -Q, \qquad Q = Q^T > 0, \quad P = P^T > 0 \\ C &= B^T P\end{aligned}$$

then the adaptation law

$$\frac{d\widehat{\theta}}{dt} = -\gamma\underbrace{B^T Px_e}_{e}\underbrace{ku^c}_{\phi} = -\gamma\phi e, \qquad \gamma = \mu k$$

will render the Lyapunov function negative definite with respect to x_e, that is

$$\begin{aligned}\frac{dV(x_e, \widetilde{\theta})}{dt} &= \frac{1}{2}x_e^T(PA + A^T P)x_e \\ &= -\frac{1}{2}x_e^T Qx_e < 0, \qquad \|x_e\| \neq 0 \\ \frac{d\widetilde{\theta}}{dt} &= \frac{d\widehat{\theta}}{dt} = -\gamma\phi e\end{aligned}$$

Whereas it is possible to claim asymptotic stability of the error dynamics with respect to the error dynamics of x_e, only stability (in the sense of Lyapunov) can be established for the adaptation error dynamics of $\widetilde{\theta}$

2. A process is modeled by

$$y(k) = b_0 u(k) + b_1 u(k-1) + e(k),$$

where $e(k)$ is a normally distributed white noise process.

a. Derive a least-squares estimator for the process. (2 p)

b. Derive expressions for the estimation error and estimation error covariance. (2 p)

c. Present an input sequence $u(k)$ resulting in a consistent estimator. Prove your claim. (1 p)

Solution

a. The standard LS-estimator is given by

$$\hat{\theta} = (\Phi_N^T \Phi_N)^{-1} \Phi_N^T Y_N$$

Here N indicates that data has ben aquired at times $1, \ldots, N$. We write the process on the form

$$y(k) = \varphi^T(k-1)\theta + e(t),$$

where $\varphi^T = [u(k)\ u(k-1)]$ and $\theta^T = [b_0\ b_1]$. The set of regressor vectors for N measurements are collected in the matrix

$$\Phi_N = \begin{bmatrix} u(1) & u(0) \\ \vdots & \vdots \\ u(N) & u(N-1) \end{bmatrix}$$

and the measured output signal is collected in

$$Y_N = \begin{bmatrix} y(1) \\ \vdots \\ y(N) \end{bmatrix}.$$

b. The estimate, given a seqence of length $N-1$ is given by

$$\hat{\theta} = (\Phi_N^T \Phi_N)^{-1} \Phi_N^T Y_N = (\Phi_N^T \Phi_N)^{-1} \Phi_N^T (\Phi_N \theta + e).$$

Hence, the corresponding estimation error is

$$\hat{\theta} - \theta = (\Phi_N^T \Phi_N)^{-1} \Phi_N^T e$$

and the error covariance becomes

$$\begin{aligned} E\{(\hat{\theta}-\theta)(\hat{\theta}-\theta)^T\} &= E\{((\Phi_N^T \Phi_N)^{-1} \Phi_N^T e) e^T \Phi_N (\Phi_N^T \Phi_N)^{-1}\} \\ &= (\Phi_N^T \Phi_N)^{-1} \Phi_N^T E\{ee^T\} \Phi_N (\Phi_N^T \Phi_N)^{-1} \\ &= \sigma_e^2 \cdot (\Phi_N^T \Phi_N)^{-1}. \end{aligned}$$

c. Supported by the theory, we choose an input signal or persisten excitation order ≥ 2. E.g., evaluation of the covariance matrix for $u(k) = (-1)^k$ yields

$$\begin{aligned} \left(\Phi_N^T \Phi_N\right)^{-1} &= \begin{bmatrix} \sum_{k=1}^N u^2(k) & \sum_{k=1}^N u(k)u(k-1) \\ \sum_{k=1}^N u(k)u(k-1) & \sum_{k=1}^N u^2(k-1) \end{bmatrix}^{-1} \\ &= \begin{bmatrix} N & -N \\ -N & N \end{bmatrix}^{-1} = \frac{1}{2N} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \end{aligned}$$

I.e.,

$$\lim_{N \to \infty} \left(\Phi_N^T \Phi_N\right)^{-1} = 0.$$

3. This problem deals with MRAC design of an STR, similar to that of homework assignment 2. The sampled process and reference models are given by

$$G = \frac{B}{A}, \quad G_m = \frac{B_m}{A_m},$$

where $\deg A = \deg A_m = 2$ and $\deg B = \deg B_m = 1$. Also, A, A_m are chosen monic. The controller structure is given by the ARMAX controller

$$Ru = -Sy + Tu^c,$$

where u^c, u and y are reference, control signal and system output, respectively.

a. Assume that the zero of B is poorly damped. Mention a negative consequency of canceling it by a controller pole. Show why it is not possible to avoid this cancellation for an arbitrary choice of B_m. (2 p)

b. Let $B = B_m$ and show that it is generally impossible to find a controller without zero cancellation where $\deg R = 0$. (2 p)

c. Describe how the controller structure can be modified in order to introduce integral action and how this affects the minimal degree solution. (2 p)

d. What is the difference between direct and indirect MRAC? (1 p)

Solution

a. Cancellation of a poorly damped process zero introduces a poorly damed controller pole resulting in a ringing control signal.

In order to avoid zero cancellation we must choose $B^+ = 1$, $B^- = B$ when factoring $B = B^- B^+$ and include B^- as a factor in B_m. Hence

$$B_m = B^- B_m^1 = B B_m^1,$$

for some polynomial B_m^1. Inserting B and B_m into the above equation yields

$$b_{m_1} z + b_{m_2} = (b_1 z + b_2) B_m^1,$$

which only has polynomial solution B_m^1 if

$$\frac{b_{m_1}}{b_{m_2}} = \frac{b_1}{b_2}.$$

Hence it is generally not possible to meet the model mathcing specification without zero cancellation.

b. If there exists a solution with $\deg R = 0$, causality of the controller implies $\deg S = 0$. From the controller structure it evident that R can always be chosen monic. I.e., we want to solve the DAB

$$\underbrace{(z^2 + a_1 z + a_2)}_{A} \underbrace{1}_{R} + \underbrace{(b_1 z + b_2)}_{B} \underbrace{s_0}_{S} = A_o \underbrace{(z^2 + a_{m_1} z + a_{m_2})}_{A_m} \underbrace{1}_{B^+}.$$

Degree matching yields $\deg A_o = 0$ and we introduce $A_o(z) = a_o$. Since the left hand side polynomial is monic we have to choose $a_o = 1$. Matching equal powers of z yields

$$\begin{cases} z^2: & 1 &= 1 \\ z^1: & a_1 + b_1 s_0 &= a_{m_1} \\ z^0: & a_2 + b_2 s_0 &= a_{m_2}, \end{cases}$$

which clearly lacks solution if e.g., $b_1 = 0,\ a_1 \neq a_{m_1}$

c. Integral action is introduced by letting $R_i = (z-1)$ be a factor of R. We hence need to solve the DAB

$$AR_iR_1 + B^- S = A_o A_m.$$

cf. Johansson (9.7), (9.8). Generally, introduction of $z-1$ as factor in R increases the minimal degree solution.

d. In in-direct MRAC STR design, process parameters are explicitly estimated and controllers are iteratively computed from these estimates. In direct MRAC STR design, the regression model is changed, so that the regressors express the controller parameters directly.

4. The dynamics of a plant are described by

$$\begin{cases} x_{k+1} &= \Phi x_k + \Gamma u_k + d_k \\ y_k &= C x_k, \end{cases}$$

with $\Phi = \frac{1}{2}$, $\Gamma = 1$ and $C = 2$. The disturbance d_k is constant $d_k = 3$.

a. Extend the state to include the disturbance state $d_k = d$ and give the extended dynamics. (2 p)

b. Explain (briefly) why a state observer is needed to use this model for control synthesis, assuming d is unknown and not directly measurable. (1 p)

c. Give the questions for a one-step-ahead linear state estimator, which placed all poles of the error dynamics at $-\frac{1}{2}$. (2 p)

Solution

a. Introducing the extended state

$$x_k^e = \begin{bmatrix} x_k \\ d_k \end{bmatrix},$$

results in the dynamics

$$\begin{cases} x_{k+1}^e &= \Phi^e x_k^e + \Gamma^e u_k \\ y_k &= C^e x_k^e, \end{cases}$$

where

$$\Phi^e = \begin{bmatrix} \Phi & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & 1 \end{bmatrix}, \Gamma^e = \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C^e = \begin{bmatrix} C & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \end{bmatrix}.$$

b. The state d is not directly measurable, but relevant when syntesizing a controller. However, it is observable, since the observability matrix

$$\begin{bmatrix} C^e \\ C^e\Phi^e \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$$

has full rank.

c. Introducing estimator dynamic

$$\hat{x}^e_{k+1|k} = \Phi^e \hat{x}^e_{k|k-1} + \Gamma^e u_k + K(y_k - C^e \hat{x}_{k|k-1}),$$

we obtain error dynamics

$$x_{k+1} - \hat{x}^e_{k+1|k} = (\Phi^e x_k + \Gamma^e u_k) - \left(\Phi^e \hat{x}^e_{k|k-1} + \Gamma^e u_k + K(y_k - C^e \hat{x}_{k|k-1})\right).$$

Denoting the estimation error $\tilde{x}^e_k = x^e_k - \hat{x}^e_{k|k-1}$, the error dynamics can be rewritten

$$\tilde{x}^e_{k+1} = (\Phi - KC)\tilde{x}^e_k,$$

where

$$\Phi - KC = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \begin{bmatrix} 2 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - 2k_1 & 1 \\ -2k_2 & 1 \end{bmatrix},$$

with

$$\det(zI - (\Phi - KC)) = z^2 + (2k_1 - \tfrac{3}{2})z + (2k_2 - 2k_1 + \tfrac{1}{2}) = (z - \tfrac{1}{2})^2 = z^2 + z + \tfrac{1}{4}$$

according to design specifications. Equating equal powers of z results in

$$\begin{cases} z^2: & 1 = 1 \\ z^1: & 2k_1 - \frac{2}{3} = 1 \\ z^0: & 2k_2 - 2k_1 + \frac{1}{2} = \frac{1}{4} \end{cases} \Leftrightarrow \begin{cases} k_1 = \frac{5}{4} \\ k_2 = \frac{9}{8} \end{cases}.$$

5. Model Predictive Control (MPC) is based on the *receding horizon* principle, illustrated in Fig. 1. The aim is to decide a number of future input values given a prediction of a finite number of future outputs. The first input value is implemented, and the procedure is repeated at the next sampling instance.

The controller is obtained by minimizing a cost function:

$$V(U_t, Y_t) = Y_t^T Q_y Y_t + U_t^T Q_u U_t \tag{1}$$

where U_t and Y_t are sequences of future control signals and outputs up to horizons N and M respectively:

$$U_t = \begin{bmatrix} u(t+N-1) \\ \vdots \\ u(t) \end{bmatrix}, \qquad Y_t = \begin{bmatrix} \hat{y}(t+M|t) \\ \vdots \\ \hat{y}(t+1|t) \end{bmatrix}$$

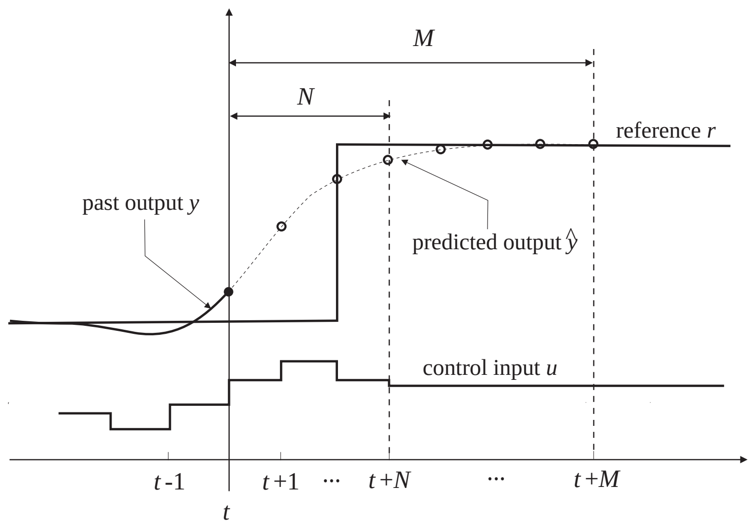


Figure 1 Illustration of the receding horizon principle used in Model Predictive Control

When the system is known, the predicted future outputs are given by the predictor:

$$\begin{bmatrix} \hat{y}(t+M|t) \\ \vdots \\ \hat{y}(t+1|t) \end{bmatrix} = \begin{bmatrix} CA^M \\ \vdots \\ CA \end{bmatrix} \hat{x}(t|t) + \begin{bmatrix} CB & CAB & CA^2B & \dots \\ 0 & CB & CAB & \dots \\ \vdots & \ddots & \ddots & \vdots \end{bmatrix} \begin{bmatrix} u(t+M-1) \\ \vdots \\ u(t+N-1) \\ \vdots \\ u(t) \end{bmatrix}$$

$$Y_t = D_x \hat{x}(t|t) + D_u U_t$$

a. Show that the cost function (1) can be written as:

$$V(U_t) = \hat{x}(t|t)^T Q \hat{x}(t|t) + U_t^T R U_t + 2\hat{x}(t|t)^T S U_t$$

and that the minimum is attained for:

$$U_t = -R^{-1} S \hat{x}(t|t)$$

(2 p)

b. The MPC formulation described here assumes that a process model is available. Can you suggest a way of modifying the algorithm to create an adaptive MPC controller? (Hint: Consider the sequence of predicted outputs Y_t, as well as the way in which process parameters are identified in the least-squares algorithm) (1 p)

Solution

a. Substitution of the predicted future outputs $Y_t = D_x \hat{x}(t|t) + D_u U_t$ into the cost function (1) gives:

$$\begin{aligned} V(U_t) &= (D_x \hat{x}(t|t) + D_u U_t)^T Q_y (D_x \hat{x}(t|t) + D_u U_t) + U_t^T Q_u U_t \\ &= \hat{x}(t|t)^T \underbrace{D_x^T Q_y D_x}_{Q} \hat{x}(t|t) + 2\hat{x}(t|t)^T \underbrace{D_x^T Q_y D_u}_{S} U_t + U_t^T \underbrace{(Q_u + D_u^T Q_y D_u)}_{R} U_t, \end{aligned}$$

assuming Q_y symmetric. To find the minimum of the cost function, take the derivative with respect to U_t and set this to zero: $\frac{\partial V(U_t)}{\partial U_t} = 0$. This gives:

$$U_t = -R^{-1}S\widehat{x}(t|t),$$

assuming symmetric R, S (which follows from symmetric Q_u, Q_y).

b. In the MPC formulation presented in the question, a state space model of the system is used to build a predictor. In adaptive control, a model is not available, and must typically be estimated. In the case of MPC, it is sufficient to obtain a prediction of future outputs from measured data. Least squares methods involve minimizing the error between a predicted output $\hat{y}(t)$ and the measured output $y(t)$. This could be extended to the estimation of prediction matrices D_x and D_u, giving $Y_t = \hat{D}_x\widehat{x}(t|t) + \hat{D}_u U_t$, producing an MPC controller for an unknown system.

6. One possible strategy for Iterative Learning Control (ILC) is given by the equations:

$$\begin{aligned} y_k(t) &= G_c(q)u_k(t) \\ e_k(t) &= r(t) - y_k(t) \\ u_k(t) &= Q(q)[u_{k-1}(t) + L(q)e_{k-1}(t)] \end{aligned}$$

where $G_c(q)$ is the closed-loop transfer function of the system and q is the forward time shift operator.

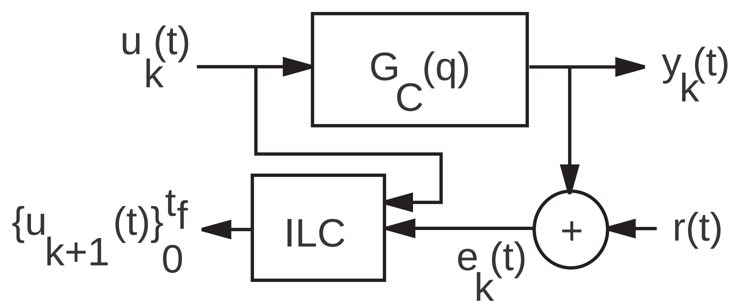


Figure 2 ILC set-up in Problem 6

a. Explain the principle of operation of Iterative Learning Control. (1 p)

b. Assume that $Q(q) = 1$ and that

$$G_C(q) = \frac{1}{(q-0.7)(q-0.9)}, \qquad L(q) = k(q-0.5)(q-0.7)(q-0.9)$$

where k is a positive constant. Does there exist $k > 0$ for which the ILC scheme converges? Motivate your answer. (2 p)

Solution

a. ILC can be used to improve tracking performance for systems in which the same reference trajectory is used repetitively. The strategy is based on collection of a data set and filtering operations upon the data. Non-causal filtering may be used since the filtering is performed offline. A typical example of an ILC application is trajectory following for a robotic manipulator, where modelling inaccuracies typically give rise to tracking errors.

b. We obtain the following recursive expression for the tracking error

$$e_k(t) = (1 - LG_C)e_{k-1}(t)$$

and convergence will be achieved if

$$\sup_{\omega h \in [-\pi, \pi]} \mid 1 - L(e^{i\omega h})G_C(e^{i\omega h}) \mid < 1 \Leftrightarrow \sup_{\omega h \in [-\pi, \pi]} \mid 1 - k(e^{i\omega h} - 0.5) \mid < 1.$$

Introducing $x = \omega h$ we have

$$\begin{aligned} & \mid 1 - k(\cos(x) + i\sin(x) - 0.5) \mid \\ = & \mid 1 - k\cos(x) - 0.5k + ik\sin(x) \mid \\ = & \sqrt{1 + k^2\cos^2(x) + k^2\sin^2(x) - k(k+2)\cos(x) + k + 0.25k^2}. \end{aligned}$$

Using the trigonometric identity we obtain

$$\sqrt{1 + k^2 - k(k+2)\cos(x) + k + 0.25k^2}$$

and the supremum is achieved when $\cos(x) = -1$, yielding

$$\sqrt{1 + k^2 + k(k+2) + k + 0.25k^2} = \sqrt{1 + 2.25k^2 + 2k},$$

which is clearly larger than 1 for all positive k. Hence, convergense is not achieved for any positive k.