Solutions to Predictive Control exam, October 16, 2007

1.

a. A regression model can be defined:

$$\hat{y}_k = \phi_{k-1}^T \theta$$

where:

$$\phi_{k-1}^T = \begin{bmatrix} -y_{k-1} & -y_{k-2} & u_{k-1} & u_{k-2} \end{bmatrix}$$
$$\theta = \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix}$$

Using Least Squares, an estimate of θ is given by:

$$\hat{\theta} = (\Phi^T \Phi)^{-1} \Phi^T \mathcal{Y}$$

b. For online use, the *Recursive Least Squares* (RLS) algorithm can be used:

$$\begin{aligned}
\hat{\theta}_k &= \hat{\theta}_{k-1} + K_k \epsilon_k \\
\epsilon_k &= y_k - \phi_{t-1}^T \hat{\theta}_{t-1} \\
K_k &= P_{k-1} \phi_{k-1} (1 + \phi_{k-1}^T P_{k-1} \phi_{k-1})^{-1} \\
P_k &= (I - K_k \phi_{k-1}^T) P_{k-1}
\end{aligned}$$

The estimate $\hat{\theta}$ is updated at each sample. To start the algorithm, an initial parameter guess θ_0 and corresponding covariance matrix P_0 must be supplied.

c. When the unknown parameters are time varying, it is desireable to disregard 'old' information, related to previous values of the parameters. This can be accomplished by the following modification to the RLS algorithm:

$$\begin{aligned}
\hat{\theta}_k &= \hat{\theta}_{k-1} + K_k \epsilon_k \\
\epsilon_k &= y_k - \phi_{t-1}^T \hat{\theta}_{t-1} \\
K_k &= P_{k-1} \phi_{k-1} (\lambda + \phi_{k-1}^T P_{k-1} \phi_{k-1})^{-1} \\
P_k &= (I - K_k \phi_{k-1}^T) P_{k-1} / \lambda
\end{aligned}$$

where λ is known as the 'forgetting factor', $\lambda \in [0, 1]$. It provides an exponentially decreasing weight on old data, meaning more recent data is primarily used for generating new estimates. This allows the estimator to 'track' time varying parameters more easily.

2.

a. The receding horizon principle is illustrated in Figure 1. The aim is to decide a number of future input values (changes) given a prediction of a finite number of predicted future outputs. This is normally done by setting up a cost function. If constraints are present, a constrained optimization problem results, which is typically solved by numerical methods. The solution is a sequence of control moves, of which the first value is implemented, and the procedure is repeated at the next sampling instant.

The prediction horizon M is the number of time steps for which predicted outputs are computed. It should be chosen such that the important time constants of the system are captured in the horizon. The control horizon N is the number of future control moves considered. Typically, this is chosen to be smaller than the prediction horizon, since the number of control moves determines the complexity of the optimization problem to be solved.

b. The constraint must be fulfilled at all time instances k, so we have:

$$-10 \leq y_{k+1} \leq 10$$

but:

$$y_{k+1} = 5x_{k+1} = 2.5x_k + 10u_k$$

Rewriting u_k as $u_{k-1} + \Delta u_k$ gives:

$$y_k = 2.5x_k + 10u_{k-1} + 10\Delta u_k$$

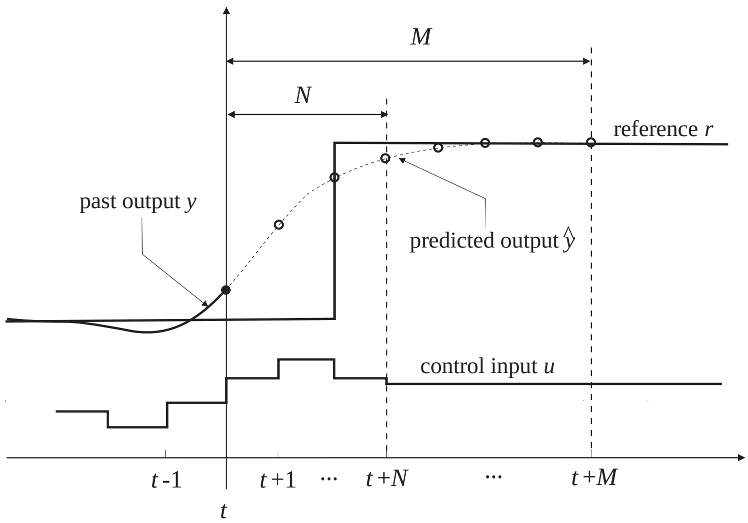
Substituting this into the constaint gives:

$$-10 \leq 2.5x_k + 10u_{k-1} + 10\Delta u_k \leq 10$$

Using the fact that $x_k = 1$ and $u_{k-1} = -1$ gives:

$$-0.25 \leq \Delta u_k \leq 1.75$$

as required.



Figur 1 Illustration of the receding horizon principle used in Model Predictive Control

c. Since the disturbance is constant we have $d_{k+1} = d_k$. To include the effects of the disturbance in the process model we can extend the state vector:

$$x_k^e \begin{bmatrix} x_k \\ d_k \end{bmatrix}$$

which allows an augmented model to be constructed:

$$\begin{bmatrix} x_{k+1} \\ d_{k+1} \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \begin{bmatrix} x_k \\ d_k \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k$$
$$y_k = [\,C \quad 0\,] \begin{bmatrix} x_k \\ d_k \end{bmatrix}$$

Since the extended state vector contains the unknown disturbance, a state observer will be needed. A Kalman filter can be used. A stable observer will converge to the correct value of d_k and the closed loop system will exhibit error-free tracking.

3.

a. The process zero is $B = B^+B^-$, where:

$$B^+ = q + \frac{b_1}{b_0} \qquad B^- = b_0$$

To cancel B^+, the R polynomial must contain B^+. Integral action is also required, so R must also contain a factor of $(q-1)$. This gives:

$$R = B^+R' \qquad R' = (q-1)R''$$

The Diophantine equation is then:

$$A(q-1)R'' + B^-S = A_mA_o$$

To determine the order of the observer polynomial A_o, the following causality condition may be used:

$$\deg A_o = 2\deg A - \deg A_m - \deg B^+ + 1 - 1 = 4 - 2 - 2 + 1 - 2 = 1$$

The degree of R'' is therefore:

$$\deg R'' = \deg A_o + \deg A_m - \deg A - 1 = 0$$

and:

$$\deg R = \deg S = \deg T = 2$$

The Diophantine equation may then be solved with this choice of degree:

$$(q^2 + a_1q + a_2)(q-1) + b_0(s_0q^2 + s_1q + s_2) = (q^2 + a_{m_1}q + a_{m_2})(q + a_o)$$

Identification of coefficients gives:

$$\begin{aligned} q^3 &: & 1 &= 1 \\ q^2 &: & a_1 - 1 + b_0s_0 &= a_o + a_{m_1} \\ q^1 &: & a_2 - a_1 + b_0s_1 &= a_{m_1}a_o + a_{m_2} \\ q^0 &: & -a_2 + b_0s_2 &= a_{m_2}a_o \end{aligned}$$

which gives

$$
\begin{aligned}
s_0 &= \frac{a_o + a_{m_1} - a_1 + 1}{b_0} \\
s_1 &= \frac{a_{m_1} a_o + a_{m_2} - a_2 + a_1}{b_0} \\
s_2 &= \frac{a_{m_2} a_o + a_2}{b_0}
\end{aligned}
$$

the closed loop system is given by

$$
\frac{BT}{AR + BS} = \frac{BT}{A_m A_o B^+} = \frac{b_0 T}{A_m A_o} = \frac{B_m}{A_m} \quad \Rightarrow \quad T = \frac{1}{b_0} B_m A_o
$$

which gives

$$
\begin{aligned}
R(q) &= q(q-1) \\
S(q) &= s_0 q^2 + s_1 q + s_2 \\
T(q) &= \frac{1}{b_0}(b_{m_1} q + b_{m_2})(q + a_o)
\end{aligned}
$$

b. Begin by letting the Diophantine equation operate on $y(t)$:

$$
ARy(t) + BSy(t) = A_m A_o B^+ y(t)
$$

Assuming cancellation of B^+, and by substituting $Ay(t) = Bu(t)$, we obtain:

$$
b_0(Ru(t) + Sy(t)) = A_m A_o y(t)
$$

This can be rewritten as:

$$
\begin{aligned}
y(t) &= \frac{b_0}{A_m A_o}(Ru(t) + Sy(t)) \\
&= \tilde{R} \underbrace{\frac{b_0}{A_m A_o} u(t)}_{u_f(t)} + \tilde{S} \underbrace{\frac{b_0}{A_m A_o} y(t)}_{y_f(t)}
\end{aligned}
$$

where $\tilde{R} = b_0 R$ and $\tilde{S} = b_0 S$. Thus a regression model can be constructed with $u_f(t)$ and $y_f(t)$ as regressors and the controller coefficients as parameters. Notice that this design assumes that the process zeros are stable.

4.

a. The Diophantine equation is

$$
(1 + 0.7z^{-1}) \quad = \quad (1 - 0.9z^{-1})(f_0 + f_1 z^{-1}) + z^{-2} g_0
$$

with the solution

$$
\begin{cases}
z0: & 1 = f_0 \\
z^{-1}: & 0.7 = -0.9 f_0 + f_1 \\
z^{-2}: & 0 = -0.9 f_1 + g_0
\end{cases}
\qquad
\begin{cases}
f_0 = 1 \\
f_1 = 1.6 \\
g_0 = 1.44
\end{cases}
$$

gives the expansion

$$y_{k+2} = F^*(q^{-1})e_{k+2} + \frac{G^*(q^{-1})}{A^*(q^{-1})}w_k$$

The resulting two-step predictor is

$$\begin{aligned}
y_{k+2} &= F^*(q^{-1})e_{k+2} + \frac{G^*(q^{-1})}{A^*(q^{-1})}w_k \\
\widehat{y}_{k+2} &= \frac{G^*(q^{-1})}{A^*(q^{-1})}w_k = \frac{G^*(q^{-1})}{C^*(q^{-1})}y_k \\
&= \frac{1.44}{1+0.7q^{-1}}y_k \\
\widehat{y}_{k+2} &= -0.7\widehat{y}_k + 1.44y_k
\end{aligned}$$

The error covariance is

$$\{\widetilde{y}^2_{k+2}|\mathcal{F}_k\} = (f_0^2 + f_1^2)\sigma 2 = 3.56\sigma 2$$

b. Using diophantine equation $C^* = A^*F^* + z^{-2}G^*$ yields the same F^* and $G*$ polynomials as in the predictor case above. This gives the minimum variance controller:

$$u(k) = \frac{G^*(q^{-1})}{B^*(q^{-1})F^*(q^{-1})}y(k) = \frac{1.44}{(1+0.5q^{-1})(1+1.6q^{-1})}y(k)$$

5. Assume that the transfer function $G(s)$ has a state-space realization

$$\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx, \qquad Y(s) = G(s)U(s)
\end{aligned}$$

and

$$\begin{aligned}
\dot{x}_m &= Ax_m + B(k_m u^c) \\
y &= Cx_m, \qquad Y^m(s) = G(s)k_m U^c(s)
\end{aligned}$$

The error model

$$\begin{aligned}
x_e &= x - x_m \\
e &= y - y_m, \qquad E(s) = G(s)(k\theta - k_m)U^c(s)
\end{aligned}$$

with the error dynamics

$$\begin{aligned}
\dot{x}_e &= Ax_e + B(k\theta - k_m)u^c = Ax_e + B\underbrace{ku^c}_{\phi}\widetilde{\theta} \\
e &= Cx_e
\end{aligned}$$

Introduce the Lyapunov function candidate

$$V(x_e, \widetilde{\theta}) = \frac{1}{2}x_e^T P x_e + \frac{\mu}{2}\widetilde{\theta}^T\widetilde{\theta}, \quad P = P^T > 0, \mu > 0$$

with the derivative

$$\begin{aligned}\frac{dV(x_e,\widetilde{\theta})}{dt} &= \frac{1}{2}x_e^T(PA+A^TP)x_e + x_e^TPB(k\theta - k_m)u^c + \mu\widetilde{\theta}^T\frac{d\widetilde{\theta}}{dt}\\ &= \frac{1}{2}x_e^T(PA+A^TP)x_e + \widetilde{\theta}^T(B^TPxku^c + \mu\frac{d\widetilde{\theta}}{dt})\end{aligned}$$

Under the conditions of the Kalman-Yakubovich-Popov (KYP) Lemma, we have for an SPR transfer function $G(s)$

$$\begin{aligned}PA + A^TP &= -Q, \qquad Q = Q^T > 0, \quad P = P^T > 0\\ C &= B^TP\end{aligned}$$

then the adaptation law

$$\frac{d\widehat{\theta}}{dt} = -\gamma\underbrace{B^TPx_e}_{e}\underbrace{ku^c}_{\phi} = -\gamma\phi e, \qquad \gamma = \mu k$$

will render the Lyapunov function negative definite with respect to x_e, that is

$$\begin{aligned}\frac{dV(x_e,\widetilde{\theta})}{dt} &= \frac{1}{2}x_e^T(PA+A^TP)x_e\\ &= -\frac{1}{2}x_e^TQx_e < 0, \qquad \|x_e\| \neq 0\\ \frac{d\widetilde{\theta}}{dt} &= \frac{d\widehat{\theta}}{dt} = -\gamma\phi e\end{aligned}$$

Whereas it is possible to claim asymptotic stability of the error dynamics with respect to the error dynamics of x_e, only stability (in the sense of Lyapunov) can be established for the adaptation error dynamics of $\widetilde{\theta}$

6.

a. ILC can be used to improve tracking performance for systems in which the same reference trajectory is used repetitively. The strategy is based on collection of a data set and filtering operations upon the data. Non-causal filtering may be used since the filtering is performed offline. A typical example of an ILC application is trajectory following for a robotic manipulator, where modelling inaccuracies typically give rise to tracking errors.

b.

1. In this case it is likely that ILC will provide improved performance, assuming stochastic disturbances are small in magnitude.
2. In this example ILC is also likely to improve performance since an accurate model describing the fluid dynamics is most likely not available.
3. Since the process outputs are corrupted by noise, it is possible that ILC will fail to converge.
4. In this case, stochastic disturbances such as wind velocity are likely to be the dominating influence on performance. Therefore it is unlikely that ILC will improve the situation.

c. We obtain the following recursive expression for the tracking error

$$e_k(t) = (1 - LG_C)e_{k-1}(t)$$

and convergence will be achieved if

$$\mid 1 - L(e^{i\omega h})G_C(e^{i\omega h}) \mid < 1$$

where $\omega h \in [-\pi, \pi]$ and h is the sampling time—*i.e.*, the Nyquist curve of $L(z)G_C(z)$ should be contained in a region in the complex plane given by a circle with radius one centered at $z = 1$. Simplification gives

$$\mid 1 - ke^{i\omega h} - 0.5 \mid < 1, \qquad \omega h \in [-\pi, \pi]$$

from which the range of $k \in [-0.5, 0.5]$ is determined.