Solutions to Exam in FRTN10 Multivariable Control, 2015-10-29

1 a. The small gain theorem gives that the closed loop system is stable if

$$\|\Delta\| \|P(i\omega)K\|_\infty < 1$$

This is equivalent to

$$2K\|1/(1+i\omega)\|_\infty < 1$$

The largest gain value is obtained for $\omega = 0$, where $|P(0)| = 1$, which gives that $K < 1/2$.

b. The loop gain is

$$\Delta P(s)K = \frac{2K}{s+1}$$

The gain margin is infinite, since the loop gain has a phase lag of at most 90°. The closed loop is hence stable for any $K > 0$. This is an exact analysis.

In subproblem a, the stability condition was sufficient but not necessary and the uncertainty was general. Hence we should expect more conservative results there.

2. From the relation

$$\Phi_y(\omega) = G(i\omega)\Phi_u(\omega)G(-i\omega)$$

we obtain

$$G(i\omega)G(-i\omega) = \frac{2(\omega^2+100)}{\omega^2+25} = 2\frac{10+i\omega}{5+i\omega}\frac{10-i\omega}{5-i\omega}$$

Hence $G(s) = \dfrac{\sqrt{2}(s+10)}{s+5}$

3. We first note that it is not the absolute weights on the states and inputs that determine the behavior of the closed-loop system, but rather the relative difference between weights. On that note, we note that case 3 has the highest relative weight on the second state (velocity), implying a slow step response which makes the step response in B a likely candidate. The shape of the step response indicate a well-damped system with almost first-order dynamics, which is the case for the poles in I. As for comparing case 1 and 2, the latter has a higher relative weight on the first state (position) and lower weight on the velocity. Both these differences could be expected to yield a faster closed-loop step response than case 1, meaning that case 2 corresponds to the step response in C and case 1 to A. As for the poles of the closed loop system, comparing A to C, we expect the poles corresponding to C to be slightly less damped and faster as compared to those matched with A. Therefore case 2 pairs with C and II, while case 1 pairs with A and III.

It is also possible to solve the problem by brute force by computing the solutions to three different Riccati equations and then calculating the resulting poles. This requires a substantial amount of work however.

4 a. The RGA is the ratio between the open-loop gain and the closed-loop gain for each input–output combination. We have

$$\mathrm{RGA}(0) = \begin{pmatrix} k_1/k_3 & k_2/k_4 \end{pmatrix}$$

b. For the given RGA(0), u_1 should be used to control y since the (1,1) element is closer to 1.

5. The system that C is controlling is given by $P(s)D$. In order to decouple this system in stationarity, $P(s)D$ could be made diagonal by choosing for example $D = P^{-1}(0)$. The expression for D is then given by

$$D = \frac{1}{P_{11}(0)P_{22}(0) - P_{21}(0)P_{12}(0)} \begin{pmatrix} P_{22}(0) & -P_{12}(0) \\ -P_{21}(0) & P_{11}(0) \end{pmatrix}.$$

Another option is to inspect the block diagram. The interaction P_{21} can be countered by selecting $D_{21}P_{22} = -D_{11}P_{21}$. Similarly, P_{12} can be countered by selecting $D_{22}P_{12} = -D_{12}P_{11}$. Choosing $D_{11} = D_{22} = 1$ we then obtain the static decoupler

$$D = \begin{pmatrix} 1 & -P_{12}(0)/P_{11}(0) \\ -P_{21}(0)/P_{22}(0) & 1 \end{pmatrix}$$

6 a. The unstable zero at $s = 8$ implies that the achievable closed-loop speed with reasonable robustness (e.g. $M_S < 2$) is smaller than 4 rad/s. The specification of 10 rad/s can thus not be fulfilled.

b. The unstable pole at $s = 1$ implies that the closed-loop speed must be larger than 2 rad/s for reasonable robustness (e.g. $M_T < 2$). The delay of 0.1 s gives that the achievable bandwidth will be below 10 rad/s. A specification of 5 rad/s could thus probably be fulfilled.

c. For an unstable pole $s = p$ we must have $|W_T(p)| \leq 1$. This necessary condition is not fulfilled since $|W_T(3)| = \frac{3+2}{2} > 1$.

d. The algebraic constraint $\Big||S| - |T|\Big| \leq 1$ is not fulfilled in the bode diagram where $|S| > 3$ and $|T| < 0.5$ for some frequencies. The specified S and T are hence impossible to achieve.

7 a. The sub-determinants are $\frac{1}{s+2}$, $\frac{2}{(s+2)(s+3)}$ so the least common denominator is $(s+2)(s+3)$ which means the poles are -2 and -3.
Rewriting the maximal sub-determinants results in $\frac{s+3}{(s+2)(s+3)}$, $\frac{2}{(s+2)(s+3)}$,which have no common zeros. This means that the system has no multivariable zeros.

b. There are several solutions to this, two of them are

1. Alt 1:

$$\begin{pmatrix} \frac{1}{s+2} & \frac{2}{(s+2)(s+3)} \end{pmatrix} = \begin{pmatrix} \frac{1}{s+2} & \frac{2}{s+2} - \frac{2}{s+3} \end{pmatrix} = \frac{1}{s+2}\begin{pmatrix} 1 & 2 \end{pmatrix} + \frac{1}{s+3}\begin{pmatrix} 0 & -2 \end{pmatrix}$$

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix},\ B = \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix},\ C = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

2. Alt 2:

$$\begin{pmatrix} \frac{1}{s+2} & \frac{2}{(s+2)(s+3)} \end{pmatrix} = \frac{1}{s+2}\begin{pmatrix} 1 & \frac{2}{s+3} \end{pmatrix}$$

$$A = \begin{pmatrix} -2 & 2 \\ 0 & -3 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

c. Yes! It is clear that the first state does not affect the output, so it is sufficient to look at the subsystem

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -2 & 2 \\ 0 & -3 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x \end{aligned}$$

which is equivalent to Alt 2 above. Alternatively, computing $C(sI - A)^{-1}B$ results in the transfer function $G(s)$.

8 a. This is shown by straightforward calculations after inserting the gramians in the corresponding Lyapunov equations:

$$A^T S + SA + BB^T = 0$$
$$A^T O + OA + C^T C = 0$$

b. The Hankel singular values are the square root of the eigenvalues of the matrix

$$SO = \begin{pmatrix} 1.25 & 0 \\ 0 & 10 \end{pmatrix}$$

which yield the following Hankel singular values:

$$\sigma = \begin{pmatrix} \sqrt{1.25} \\ \sqrt{10} \end{pmatrix} \approx \begin{pmatrix} 1.12 \\ 3.16 \end{pmatrix}$$

c. To find a balanced realization, we apply the transformation $\xi = Tx$ so that the new controllability and observability gramians are equal, i.e. $S_\xi = O_\xi$. To find a suitable transformation matrix T, we can use the fact that S and O are diagonal to guide is to find a diagonal T. The gramians of the transformed system can be expressed in the old gramians as $S_\xi = TST^T$ and $O_\xi = T^{-T}OT^{-1}$. This yield

$$\begin{aligned} S_\xi &= \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \begin{pmatrix} 2.5 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} = \\ &= \begin{pmatrix} 1/t_1 & 0 \\ 0 & 1/t_2 \end{pmatrix} \begin{pmatrix} 0.5 & 0 \\ 0 & 20 \end{pmatrix} \begin{pmatrix} 1/t_1 & 0 \\ 0 & 1/t_2 \end{pmatrix} = O_\xi \\ &\Longrightarrow \begin{cases} 2.5t_1^2 = 0.5/t_1^2 \\ 0.5t_2^2 = 20/t_2^2 \end{cases} \Longrightarrow \begin{cases} t_1 = 5^{-1/4} \\ t_2 = 40^{1/4} \end{cases} \end{aligned}$$

A balanced realization is then given by

$$\dot{\xi} = TAT^{-1}\xi + TBu = A_\xi \xi + B_\xi u$$
$$y = CT^{-1}\xi = C_\xi \xi$$

Calculations yield

$$A_\xi = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \begin{pmatrix} -1 & -4 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1/t_1 & 0 \\ 0 & 1/t_2 \end{pmatrix} = \begin{pmatrix} -1 & -4t_1/t_2 \\ 0 & -1 \end{pmatrix} \approx \begin{pmatrix} -1 & -1.0637 \\ 0 & -1 \end{pmatrix}$$

$$B_\xi = TB = \begin{pmatrix} 2t_1 & t_1 \\ t_2 & 0 \end{pmatrix} \approx \begin{pmatrix} 1.38 & 0.67 \\ 2.51 & 0 \end{pmatrix}$$

$$C_\xi = CT^{-1} = \begin{pmatrix} 1/t_1 & 2/t_2 \\ 0 & 6/t_2 \end{pmatrix} \approx \begin{pmatrix} 1.5 & 0.8 \\ 0 & 2.39 \end{pmatrix}$$