LUND INSTITUTE OF TECHNOLOGY
Lund University

Department of
AUTOMATIC CONTROL

Multivariable Control (FRTN10)

Exam October 19, 2011, hours: 8.00-13.00

Points and grades

All answers must include a clear motivation and a well-formulated answer. Answers may be given in English or Swedish. The total number of points is 25. The maximum number of points is specified for each subproblem.

Accepted aid

The textbook *Glad & Ljung*, standard mathematical tables like TEFYMA, an authorized "Formelsamling i Reglerteknik"/"Collection of Formulas" and a pocket calculator. Handouts of lecture notes and lecture slides are also allowed.

Results

The result of the exam will be posted on the notice-board at the Department. The result as well as solutions will be available on the course home page:
http://www.control.lth.se/Education/EngineeringProgram/FRTN10.html

1. Consider the system

$$y + 2\dot{y} + \ddot{y} = u_1 + 3u_2.$$

a. Find a state-space representation of the system. (2 p)

b. Find the transfer function matrix of the system. (1 p)

c. Is the system controllable? Is it observable? (1 p)

Solution

a. Define

$$x_1 = y, \; x_2 = \dot{y}.$$

Then a state-space representation is given by

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -x_1 - 2x_2 + u_1 + 3u_2.\end{aligned}$$

Written in matrix notation this becomes

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 1 & 3 \end{pmatrix} u\\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x\end{aligned}$$

b. The transfer matrix $G(s)$ is most easily found by taking the Laplace transform of (1):

$$(s^2 + 2s + 1)Y(s) = U(s)_1 + 3U(s)_2 \Rightarrow G(s) = \frac{\begin{pmatrix} 1 & 3 \end{pmatrix}}{s^2 + 2s + 1}.$$

c. The controllability matrix is

$$\begin{pmatrix} 0 & 0 & 1 & 3 \\ 1 & 3 & -2 & -6 \end{pmatrix},$$

and the observability matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Both matrices have full rank which means that the system is both controllable and observable.

2. The system

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

is chosen to be LQ-controlled using the cost function

$$J = \int_0^\infty (x_2^T Q x_2 + u^T r u)\mathrm{d}t.$$

Four responses to the initial conditions $x_0 = (5 \quad 3)^T$ are shown in Figure 1. Pair these plots to the corresponding weights shown below. Motivate.

$$\mathbf{A}: \quad Q = 1,\ r = 1 \qquad \mathbf{B}: \quad Q = 1,\ r = 0.01$$
$$\mathbf{C}: \quad Q = 0.01,\ r = 1 \qquad \mathbf{D}: \quad Q = 1,\ r = 100$$

(3 p)

Solution

C and D will give the same signal since the weights are scaled versions of each other. Responses 2 and 4 fits into this description. B puts much smaller weight on the control signal than A, which implies that B will be faster than A. This corresponds to response 3. The answer looks like follows:

$$A - 1 \qquad B - 3$$
$$C - 2,4 \quad D - 2,4$$

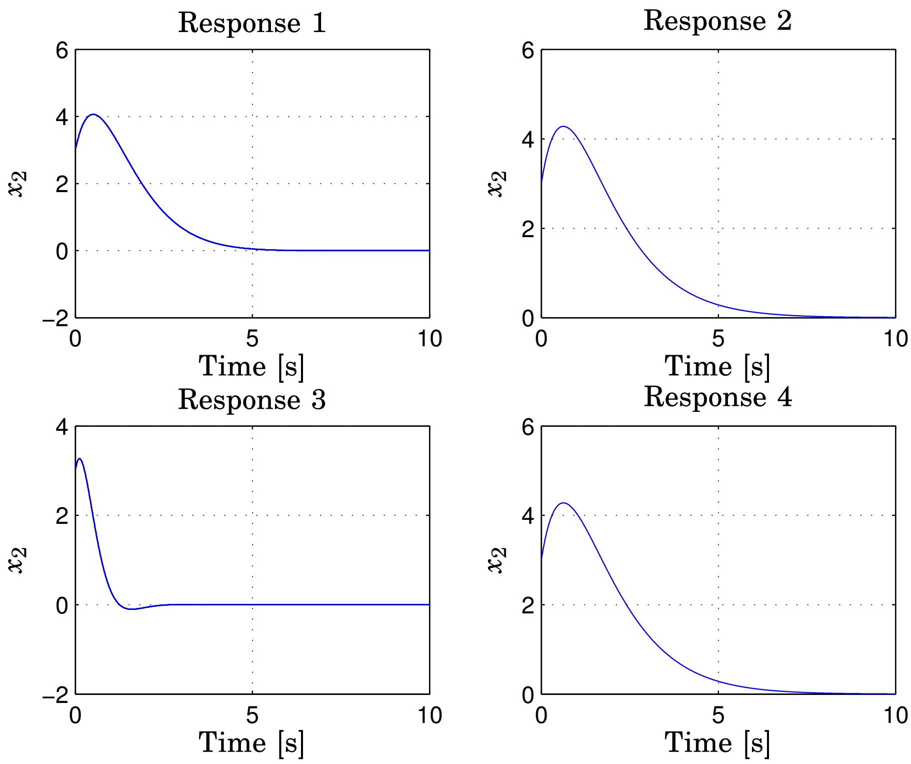

Figure 1 Figure belonging to Problem 2. Note that the diagrams do not have the identical scalings for the amplitude axes.

3. Consider the block diagram for the dynamic system in Figure 2. For perfect reference following one might like to have that the complementary sensitivity function $T(i\omega) = 1$ for all frequencies ω. However, give at least one reason for why this is in general not desirable (also in general not feasible). (1 p)

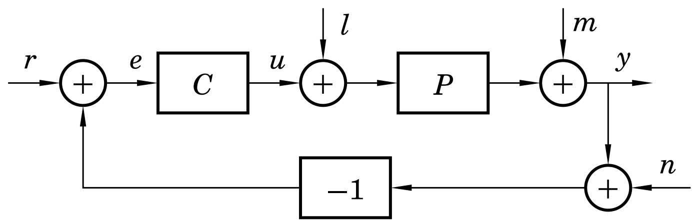

Figure 2 Figure belonging to Problem 3.

Solution

The transfer function from measurement noise to process output also equals $-T(s)$. Since measurement often has high-frequency contents, we would like $T(s)$ to be small for high frequencies. Also, $T(s)$ maps relative errors in the model to the output. Since models typically are inaccurate for high frequencies, we would again like $T(s)$ to be small for high frequencies.

4. Consider the block diagram in Figure 3. The process P is given by

$$P(s) = \frac{s-3}{(s+1)(s+2)}$$

You have been assigned the task to design the controller C such that the following specifications are fulfilled:

- The system should be internally stable.
- Good tracking of reference signals for frequencies $\omega \leq 1$ rad/s.

Your colleague claims to have solved the problem and presents you the sensitivity function in Figure 3 Would you implement your colleague's controller? Motivate your answer! (2 p)

Solution

Notice that P has a zero in 3. This zero constitutes a limitation on the achievable bandwidth. In particular $|S(i\omega)|$ cannot be small for all frequencies $\omega > 3$. The sensitivity function in Figure 3 is small for a very wide frequency range (including 100 rad/s!). This means that your colleagues design was not subject to this limitation. This can only be the case if the unstable pole was canceled by an unstable pole in the controller. But this means that $u/n = SC$ has an unstable pole. So the conclusion is: no! you should not implement the controller.

5. Consider the following transfer matrix:

$$G(s) = \begin{pmatrix} \frac{s+2}{(s+1)^2} & 0 \\ 0 & \frac{s+1}{(s+2)^2} \end{pmatrix}.$$

a. Find the poles and transmission zeros of the transfer matrix. (2 p)

b. The system is observable and controllable, which means that there are no pole-zero cancellations. Explain how this relates to your answer in **a**) and how it differs from a single input single output case. (1 p)

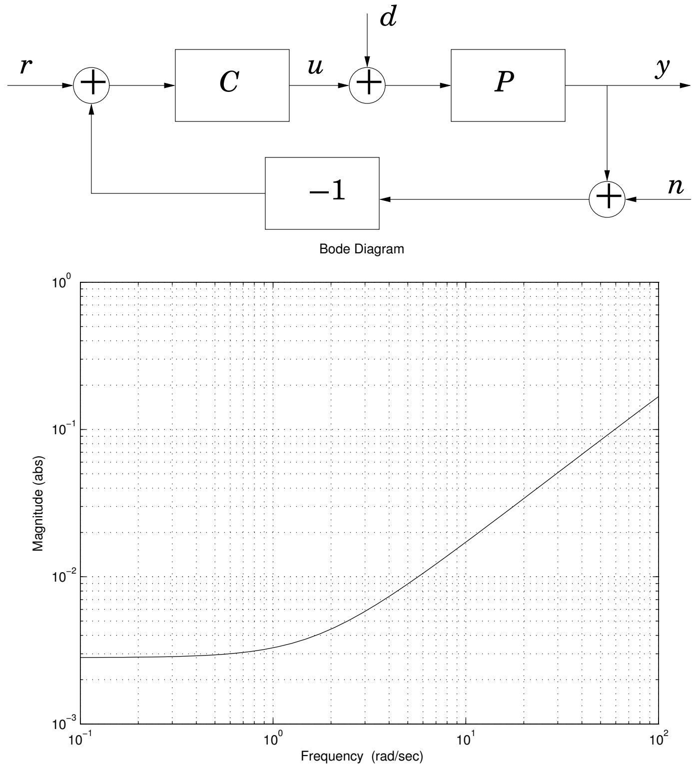

Figure 3 Block diagram and sensitivity function in Problem 4

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Solution

a. The poles of the transfer matrix is defined as the pole-polynomial's zeros, where the poly-polynomial is the least common denominator of the minors to $G(s)$ (Theorem 3.5, Glad & Ljung): The minors of the transfer matrix are

$$\frac{s+2}{(s+1)^2}, \; \frac{s+1}{(s+2)^2}, \; 0, \text{ and}$$
$$\frac{(s+1)(s+2)}{(s+2)^2(s+1)^2}.$$

Therefore the pole-polynomial is

$$(s+2)^2(s+1)^2,$$

and the poles are located in -1 and -2, with muliplicity two.

The zeros on the other hand is defined as the zeros of the zero-polynomial, which is the largest common divisor to the numerators for the maximal minors of $G(s)$ (Theorem 3.6, Glad & Ljung): From the maximal minor above, one sees that the largest common divisor is

$$(s+1)(s+2),$$

which implies that the zeros are in -1 and -2, with multiplicity one.

b. In the scalar case it is not possible to have coincident poles and zeros, since they cancel each other. In this case we have a fully decoupled system where the poles and corresponding transmission zeros appear in different channels.

6. Find a minimal state space realization of

$$G(s) = \begin{pmatrix} \frac{s+2}{s+1} & \frac{1}{s+1} \\ \frac{2s+3}{s^2+3s+2} & \frac{1}{s+1} \end{pmatrix}$$

(3 p)

Solution

First note that

$$\frac{s+2}{s+1} = 1 + \frac{1}{s+1}$$
$$\frac{2s+3}{s^2+3s+2} = \frac{2s+3}{(s+1)(s+2)} = \frac{1}{s+1} + \frac{1}{s+2}$$

This gives

$$G(s) = \begin{pmatrix} \frac{s+2}{s+1} & \frac{1}{s+1} \\ \frac{2s+3}{s^2+3s+2} & \frac{1}{s+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{s+1} & \frac{1}{s+1} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \frac{1}{s+2} & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
$$= \frac{1}{s+1}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{s+2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
$$= \frac{1}{s+1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix} + \frac{1}{s+2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

This gives

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \; D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

7. Consider the 2-DOF control setup in Figure 4. P is the plant and C_1 and C_2 are the control blocks to be designed. The objective is to make the control error $e = r - y$ small without making the control effort u too large.

a. What is the relation between the signals in the generalized plant depicted in Figure 5 and the signals in Figure 4? (1 p)

b. Derive the transfer function G in Figure 5. (2 p)

Solution

a. Regulated outputs is the error $e = r - y$ and the control signal u. Exogenous inputs are r, d and n. Signals available to the controller are r and $-(y+n)$. There is only one control signal u. In conclusion:

$$\bar{z} = \begin{pmatrix} r - y \\ u \end{pmatrix}, \quad \bar{w} = \begin{pmatrix} r \\ d \\ n \end{pmatrix}, \quad \bar{y} = \begin{pmatrix} r \\ -y - n \end{pmatrix}, \quad \bar{u} = u$$

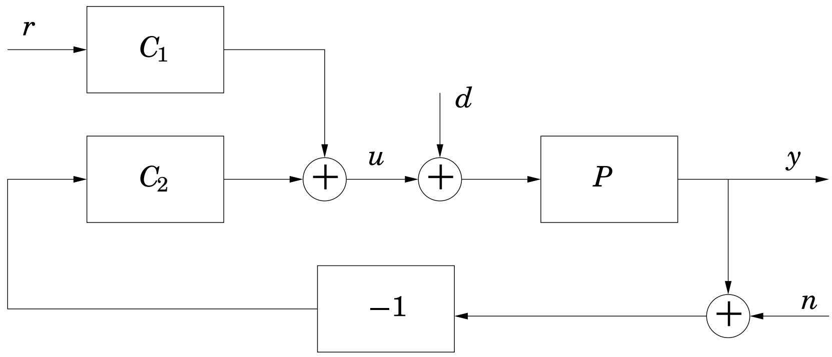

Figure 4 2-DOF control setup in problem 7

b. To derive G, remove the control blocks in Figure 4 (see Figure 6). We see that

$$\begin{aligned}
\bar{z}_1 &= r - y = r - Pd - Pu \\
\bar{z}_2 &= u \\
\bar{y}_1 &= r \\
\bar{y}_2 &= -y - n = -Pd - Pu - n
\end{aligned}$$

So

$$\begin{pmatrix} \bar{z} \\ \bar{y} \end{pmatrix} = \underbrace{\begin{bmatrix} I & -P & 0 & -P \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & -P & I & -P \end{bmatrix}}_{G} \begin{pmatrix} \bar{w} \\ \bar{u} \end{pmatrix}$$

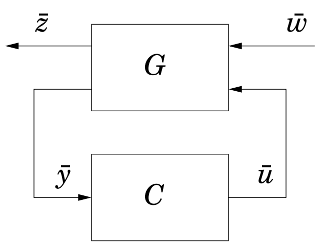

Figure 5 Generalized plant in problem 7

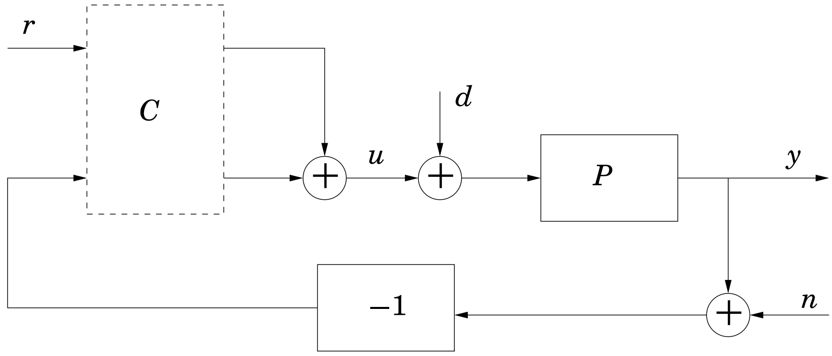

Figure 6 Setup in problem 7 with controller removed

8. This problem is about choosing noise models in LQG design. Let P be a single input single output transfer function and set

$$H(s) = \frac{\omega_0^2}{s^2 + 0.1\omega_0 s + \omega_0^2}$$

The controller C is to be determined by LQG-design. The objective is to minimize the following cost function:

$$J(u) = \int_0^\infty z^2(t) + u^2(t)dt$$

The model used in the design will be chosen as one of the models depicted in the block diagrams in Figure 7–9. The signals w_1 and w_2 are independent Gaussian white noise processes with variance 1.

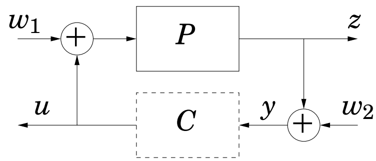

Figure 7 Model I

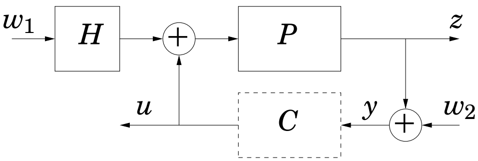

Figure 8 Model II

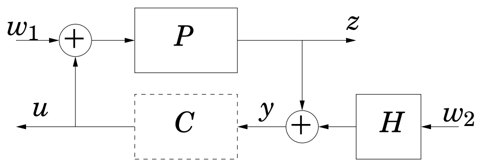

Figure 9 Model III

a. Which of the block diagrams in Figure 7–9 results in the controller with the best robustness to *measurement noise* at frequencies around ω_0. Motivate your answer! (1 p)

b. Which of the block diagrams in Figure 7–9 results in the controller with the best robustness to *load disturbance noise* at frequencies around ω_0. Motivate your answer! (1 p)

c. Which of the models in Figure 7–9 will result in the controller that gives the **largest** value of $|S(i\omega_0)P(i\omega_0)|$, where $S = \dfrac{1}{1+PC}$ is the sensitivity function.
Hint: Remember that $S + T = 1$. (2 p)

Solution

a. The weighting filter H amplifies frequencies around ω_0. By introducing H as an input weight for the measurement noise in estimation problem we have better robustness towards measurement noise. **Answer: III**

b. Similarly by introducing H as an input weight for the load disturbance noise we obtain better robustness against load disturbance noise. **Answer: II**

c. In Figure 7 we have

$$z = SPw_1 - Tw_2$$
$$u = Tw_1 + SCw_2$$

By introducing H as in III, we are penalizing $|T|$ and $|SC|$ more around ω_0, thus forcing $|C|$ to be small around this frequency. Since $S+T=1 \quad \Rightarrow$ $|T|+|S| \geq 1$ this will be at the expense of $|S|$ and thus also $|SP|$. **Answer: III**

9. Consider the system

$$\dot{x} = A_x x + B_x u = \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u$$
$$y = C_x x = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Show that the state transformation

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = Tx = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

will transform the system into a new state space form which is a *balanced realization*

$$\dot{z} = A_z z + B_z u$$
$$y = C_z z$$

by showing that the controllability Gramian S_z and the observability Gramian O_z will be the same.

Hint: To show this you don't need to calculate the explicit values of S_z and O_z, even though that is also a possibility. (2 p)

Solution

$z = Tx \implies \dot{z} = T\dot{x} = T(A_x x + B_x u) = TA_x T^{-1} z + TB_x u$ and $y = C_x x = C_x T^{-1} z$

$$A_z = TA_x T^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$B_z = TB_x = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$C_z = C_x T^{-1} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

We get the controllability Gramian S_z by solving

$$A_z S_z + S_z A_z^T + B_z B_z^T = 0$$

and the observability Gramian O_z by solving

$$A_z^T O_z + O_z A_z + C_z^T C_z = 0$$

As we can see that $C_z^T = B_z$ and $A_z = A_z^T$ we will thus have the Gramians $S_z = O_z$ without any further calculations and thus the realization is balanced.

Alternative solution:

S_z is explicitly calculated as

$$\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix} + \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Longrightarrow$

$$\begin{bmatrix} -2s_{11}+1 & -3s_{12}+1 \\ -3s_{12}+1 & -4s_{22}+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Longrightarrow S_z = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix}$ and O_z will corrispondingly get the same value (omitted here) which shows that the state space form with the z-states is a balanced realization.

Good luck!