LUND INSTITUTE OF TECHNOLOGY
Lund University

Department of
AUTOMATIC CONTROL

Multivariable Control Exam

Exam 2014-01-08

Grading

All answers must include a clear motivation and a well-formulated answer. Answers may be given in English or Swedish. The total number of points is 25. The maximum number of points is specified for each subproblem.

Accepted aid

The textbook *Glad & Ljung*, standard mathematical tables like TEFYMA, an authorized "Formelsamling i Reglerteknik"/"Collection of Formulas" and a pocket calculator. Handouts of lecture notes and lecture slides are also allowed.

Results

The results will be reported via LADOK.

1. Consider the feedback system in Figure 1.

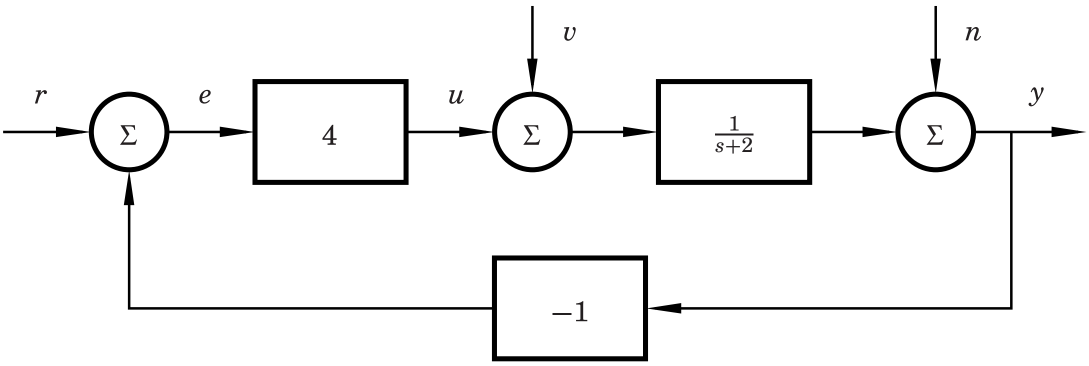

Figure 1 The system in Problem 1.

a. Give a state space realization of the system with inputs r, v and n and outputs e, u and y. (1 p)

b. Determine the entries of the transfer matrix

$$\begin{pmatrix} P_{er}(s) & P_{ev}(s) & P_{en}(s) \\ P_{ur}(s) & P_{uv}(s) & P_{un}(s) \\ P_{yr}(s) & P_{yv}(s) & P_{yn}(s) \end{pmatrix}$$

mapping the inputs r, v and n to the outputs e, u and y. (1 p)

c. After permutation (re-ordering) of the inputs and outputs, the step responses of the system have been plotted in Figure 2. Determine the new order of inputs and outputs. (1 p)

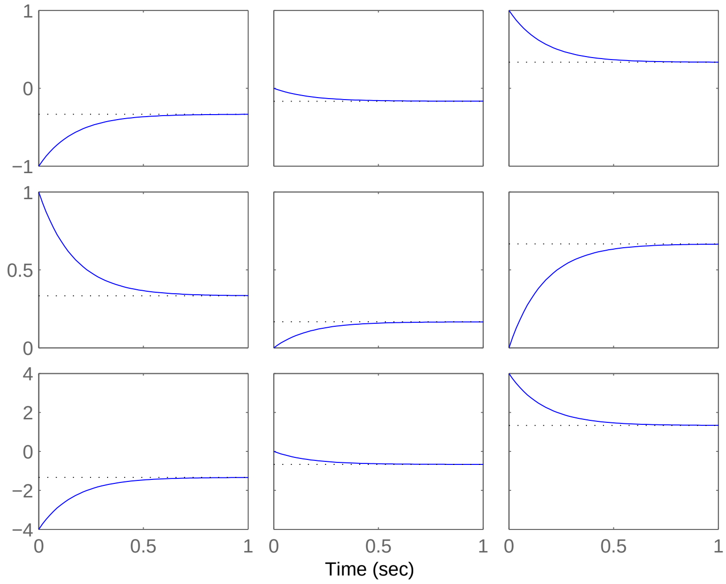

Figure 2 Step response plots for Problem 1. The inputs and outputs have been re-ordered.

d. Suppose that the load disturbance v and the measurement disturbance n are both constant and zero, while the reference signal r is a zero-mean stochastic process with spectral density $\Phi_r(\omega) = \frac{2}{\omega^2+100}$. Determine the spectral density of e. (2 p)

Solution

a. The first order process can be written

$$\dot{x} = -2x + u + v$$

Combining this with

$$u = 4e = 4(r - y) = 4[r - (x + n)] = 4r - 4x - 4n$$

gives the state realization

$$\dot{x} = \underbrace{-6}_{A} x + \underbrace{\begin{pmatrix} 4 & 1 & -4 \end{pmatrix}}_{B} \begin{pmatrix} r \\ v \\ n \end{pmatrix}$$

$$\begin{pmatrix} e \\ u \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix}}_{C} x + \underbrace{\begin{pmatrix} 1 & 0 & -1 \\ 4 & 0 & -4 \\ 0 & 0 & 1 \end{pmatrix}}_{D} \begin{pmatrix} r \\ v \\ n \end{pmatrix}$$

b. The transfer matrix is

$$\begin{aligned}
\begin{pmatrix} P_{er}(s) & P_{ev}(s) & P_{en}(s) \\ P_{ur}(s) & P_{uv}(s) & P_{un}(s) \\ P_{yr}(s) & P_{yv}(s) & P_{yn}(s) \end{pmatrix} &= C(sI - A)^{-1}B + D \\
&= \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix} (s+6)^{-1} \begin{pmatrix} 4 & 1 & -4 \end{pmatrix} + \begin{pmatrix} 1 & 0 & -1 \\ 4 & 0 & -4 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 - \frac{4}{s+6} & -\frac{1}{s+6} & -1 + \frac{4}{s+6} \\ 4 - \frac{16}{s+6} & -\frac{4}{s+6} & -4 + \frac{16}{s+6} \\ \frac{4}{s+6} & \frac{1}{s+6} & 1 - \frac{4}{s+6} \end{pmatrix}
\end{aligned}$$

c. The starting points of the step responses should be given by the D-matrix, so the input order is (n, v, r) and the output order is (e, y, u).

d. The spectrum of e is given by the formula

$$\begin{aligned}
\Phi_e(\omega) &= P_{er}(i\omega)\Phi_r(\omega)P_{er}(i\omega)^* \\
&= \left(1 - \frac{4}{i\omega+6}\right) \frac{2}{\omega^2+100} \left(1 - \frac{4}{i\omega+6}\right)^* \\
&= \frac{2(\omega^2+4)}{(\omega^2+36)(\omega^2+100)}
\end{aligned}$$

2. Your boss has heard that you are great at automatic control and wants some help finding a good controller for the process $P(s) = \frac{1}{1+s}e^{-0.5s}$.

a. He wants a fast system; the bandwidth ω_b should be at least 10 rad/s. If it is possible, find a stabilizing controller that meets the specification. If it is not possible, explain why. (1 p)

b. Your boss gets a little impatient and tries to find a controller himself. He claims that he will get a fast enough system with the controller

$$C(s) = \frac{50 + 10s}{s} e^{0.5s}.$$

He shows you the margin plot of the open loop system PC where everything looks nice, see Figure 3. You do however see a very big problem with this design. What is the problem that you need to explain to your boss? (1 p)

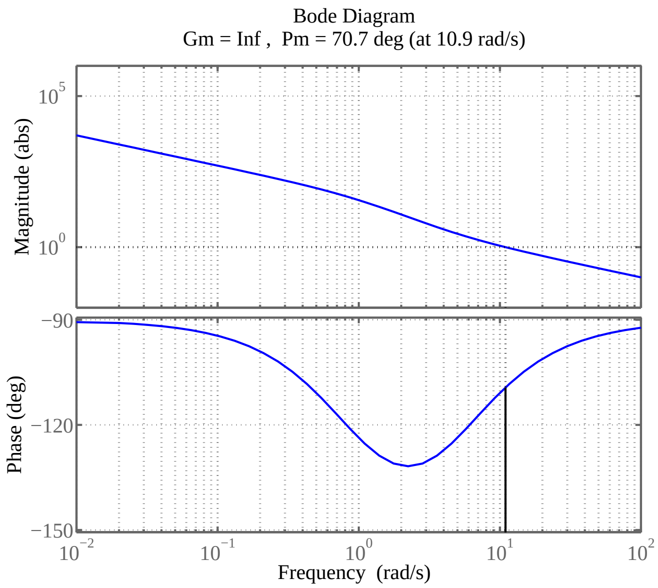

Figure 3 Margin plot for the open loop system in problem 2b.

Solution

a. It is not possible to fulfill the specification due to the fundamental limitation on the bandwidth of a time-delayed system. A system on the form $G(s) = G_1(s)e^{-sL}$, where G_1 is a rational function, will have the limited bandwidth $\omega_B < 1/L$ which in this case means that you can get a bandwidth of at most 2 rad/s.

b. You can not realize a controller with the factor $e^{0.5s}$ in it, since such a controller would be non-causal.

3. Consider the double integrator

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$
$$z = \begin{bmatrix} 1 & 2 \end{bmatrix} x.$$

a. Find the LQR controller that minimizes

$$\int_0^\infty \left(z^T(t)z(t) + u^T(t)u(t)\right) \mathrm{d}t.$$

(2.5 p)

b. Determine the stationary value $u^*(r)$ of the optimal input u when the reference value for z is r. (0.5 p)

c. Find the LQR controller that minimizes

$$\int_0^\infty \left((z(t) - r)^2 + (u(t) - u^*(r))^2\right) \mathrm{d}t.$$

(1 p)

Solution

a. The fastidious would start by ascertaining controllability of the system. Since we are not, we skip that step. Let

$$S = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}$$

be the unique positive definite solution to the Riccati equation

$$M^T Q_1 M + A^T S + SA - (SB + Q_{1,2})Q_2^{-1}(SB + Q_{1,2})^T = 0,$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad M = \begin{bmatrix} 1 & 2 \end{bmatrix},$$
$$Q_1 = I, \qquad Q_2 = I, \qquad Q_{1,2} = 0.$$

This yields

$$\begin{aligned}
&\begin{bmatrix} 1 & 2 \end{bmatrix}^T \begin{bmatrix} 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^T \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} + \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \\
&\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} = 0 \\
\iff \quad &\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ s_1 & s_2 \end{bmatrix} + \begin{bmatrix} 0 & s_1 \\ 0 & s_2 \end{bmatrix} - \begin{bmatrix} s_2 \\ s_3 \end{bmatrix} \begin{bmatrix} s_2 & s_3 \end{bmatrix} = 0 \\
\iff \quad &\begin{bmatrix} 1 - s_2^2 & 2 + s_1 - s_2 s_3 \\ 2 + s_1 - s_2 s_3 & 4 + 2s_2 - s_3^2 \end{bmatrix} = 0. \\
\iff \quad &\begin{cases} s_2 = 1, \\ s_1 = s_2 s_3 - 2, \\ s_3 = \sqrt{4 + 2s_2} \end{cases} \quad \iff \quad \begin{cases} s_2 = 1, \\ s_1 = \sqrt{6} - 2, \\ s_3 = \sqrt{6}. \end{cases}
\end{aligned}$$

This gives the optimal state feedback gain

$$L = Q_2^{-1}(SB + Q_{1,2})^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} \sqrt{6}-2 & 1 \\ 1 & \sqrt{6} \end{bmatrix} = [\,1 \quad \sqrt{6}\,].$$

Thus, the sought controller is

$$u(x) = -[\,1 \quad \sqrt{6}\,]\,x.$$

b. In stationarity we have

$$\begin{aligned} \dot{z} = M\dot{x} = \dot{x}_1 + 2\dot{x}_2 = 0 \\ \iff \dot{x}_1 = -2\dot{x}_2. \end{aligned} \tag{1}$$

In stationarity we also have $\dot{u} = 0$, which together with the second state equation yields

$$0 = \dot{u} = \dot{x}_2. \tag{2}$$

Combining the two state equations with equations (1) and (2) yields in stationarity

$$u = x_2 = \dot{x}_1 = -2\dot{x}_2 = 0.$$

The stationary input value

$$u^*(r) = 0$$

is thus necessary in order for $z = r$ in stationarity, for any r.

c. The integral is minimized with the LQR controller

$$u = -Lx + l_r r,$$

where L was determined in problem **a.**, and l_r is chosen to get unitary stationary gain in the closed-loop system, that is

$$\begin{aligned} l_r &= (M(BL-A)^{-1}B)^{-1} = \left([\,1 \quad 2\,] \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} [\,1 \quad \sqrt{6}\,] - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)^{-1} \\ &= \left([\,1 \quad 2\,] \begin{bmatrix} 0 & -1 \\ 1 & \sqrt{6} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)^{-1} = \left([\,1 \quad 2\,] \begin{bmatrix} \sqrt{6} & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)^{-1} \\ &= \left([\,1 \quad 2\,] \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)^{-1} = 1. \end{aligned}$$

Thus, the sought controller is

$$u(x,r) = -[\,1 \quad \sqrt{6}\,]\,x + r.$$

Note that the solution is entirely independent of u^*, which it would have been even if u^* were non-zero.

4.

a. Consider control of the process

$$P(s) = \frac{1}{s+1}.$$

If a proportional controller

$$C(s) = K$$

is used, for what values of K will the closed-loop system in Figure 4 be stable? (1 p)

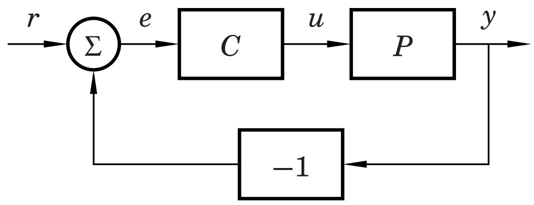

Figure 4 Block diagram of the closed-loop system in Problem 4.

b. Consider the situation where C instead is an unknown time-varying function $K(t)$, with an upper and lower limit of $\pm\alpha$. That is, for all times

$$-\alpha \leq K(t) \leq \alpha.$$

For what positive values of α can stability of the closed-loop system be guaranteed? *Hint: Use the small-gain theorem!* (3 p)

Solution

a. The closed-loop transfer function is given by

$$G(s) = \frac{P(s)C(s)}{1+P(s)C(s)} = \frac{\frac{K}{s+1}}{1+\frac{K}{s+1}} = \frac{K}{s+1+K}.$$

The closed-loop system is thus (asymptotically) stable if and only

$$K > -1.$$

b. We rewrite the block diagram into a form compatible with the small gain theorem, which yields the block diagram in Figure 5.

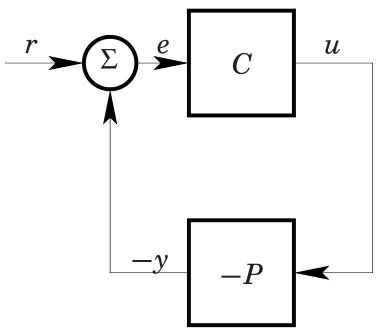

Figure 5 Block diagram of the closed-loop system in Problem 4.

Since both C and $-P$ are stable systems, we can apply the small gain theorem. Since $-P$ is linear, time-invariant and SISO, its system gain is

$$||-P||_\infty = \sup_\omega |-G(i\omega)| = \sup_\omega \left|-\frac{1}{i\omega+1}\right| = \sup_\omega \frac{1}{\sqrt{\omega^2+1}} = 1.$$

For C we have

$$||C||_\infty^2 = \sup_{e\neq 0} \frac{||K\cdot e||_2^2}{||e||_2^2} = \sup_{e\neq 0} \frac{\int_{-\infty}^{\infty}(K(t)\cdot e(t))^2 \mathrm{d}t}{\int_{-\infty}^{\infty}(e(t))^2 \mathrm{d}t} \leq \sup_{e\neq 0} \frac{\alpha^2 \int_{-\infty}^{\infty}(e(t))^2 \mathrm{d}t}{\int_{-\infty}^{\infty}(e(t))^2 \mathrm{d}t}$$
$$= \alpha^2.$$

The small gain theorem thus gives stability if

$$\begin{aligned} & ||C||_\infty \cdot ||-P||_\infty < 1 \\ \iff & ||C||_\infty < \frac{1}{||-P||_\infty} \\ \iff & \alpha < 1. \end{aligned}$$

5. Consider a system $\mathcal{S}_1$ described by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u, \\ y &= [0 \quad 2]\, x. \end{aligned}$$

a. Compute the Hankel singular values. (2 p)

b. Balanced truncation is performed to yield a first-order approximation $\mathcal{S}_2$ of the original system $\mathcal{S}_1$. Provide an upper bound on $||\mathcal{S}_1 - \mathcal{S}_2||$. Note that you do not need to calculate $\mathcal{S}_1$. (1 p)

Solution

a. Let

$$S = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}, \quad O = \begin{bmatrix} o_1 & o_2 \\ o_2 & o_3 \end{bmatrix}$$

be the controllability and observability Gramians, respectively. The controllability Gramian is found as the unique symmetric solution of the Lyapunov equation

$$\begin{aligned} & AS + SA^T + BB^T = 0 \\ \Leftrightarrow\ & \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} + \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} [2 \quad 0] = 0 \\ \Leftrightarrow\ & \begin{bmatrix} -s_1 & -s_2 \\ -s_1 - s_2 & -s_2 - s_3 \end{bmatrix} + \begin{bmatrix} -s_1 & -s_1 - s_2 \\ -s_2 & -s_2 - s_3 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} = 0 \\ \Leftrightarrow\ & \begin{cases} -2s_1 + 4 = 0, \\ -s_1 - 2s_2 = 0, \\ -2s_2 - 2s_3 = 0 \end{cases} \Leftrightarrow \begin{cases} s_1 = 2, \\ s_2 = -1, \\ s_3 = 1. \end{cases} \end{aligned}$$

The observability Gramian is found as the unique symmetric solution of the Lyapunov equation

$$
\begin{aligned}
& A^T O + OA + C^T C = 0 \\
\Leftrightarrow \quad & \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} o_1 & o_2 \\ o_2 & o_3 \end{bmatrix} + \begin{bmatrix} o_1 & o_2 \\ o_2 & o_3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \begin{bmatrix} 0 & 2 \end{bmatrix} = 0 \\
\Leftrightarrow \quad & \begin{bmatrix} -o_1 - o_2 & -o_2 - o_3 \\ -o_2 & -o_3 \end{bmatrix} + \begin{bmatrix} -o_1 - o_2 & -o_2 \\ -o_2 - o_3 & -o_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} = 0 \\
\Leftrightarrow \quad & \begin{cases} -2o_1 - 2o_2 = 0, \\ -2o_2 - o_3 = 0, \\ -2o_3 + 4 = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} o_1 = 1, \\ o_2 = -1, \\ o_3 = 2. \end{cases}
\end{aligned}
$$

The Gramians are thus

$$
S = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \quad O = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.
$$

The Hankel singular values σ are given by

$$
\begin{aligned}
0 = \det(SO - \sigma^2 I) = \begin{vmatrix} 3 - \sigma^2 & -4 \\ -2 & 3 - \sigma^2 \end{vmatrix} &= (3 - \sigma^2)^2 - 8 = \sigma^4 - 6\sigma^2 + 1 \\
\Leftrightarrow \quad & \sigma^2 = 3 \pm \sqrt{8}.
\end{aligned}
$$

Since the Hankel singular values are positive, we get

$$
\sigma = \sqrt{3 \pm \sqrt{8}} = \sqrt{3 \pm 2\sqrt{2}} = \sqrt{(1 \pm \sqrt{2})^2} = \sqrt{2} \pm 1.
$$

b. We have

$$
||\mathcal{S}_1 - \mathcal{S}_2|| = \sup_{u \neq 0} \frac{||\mathcal{S}_1(u) - \mathcal{S}_2(u)||_2}{||u||_2} \leq \sup_{u \neq 0} 2\sigma_r = 2\sigma_r,
$$

where σ_r is the Hankel singular value of the truncated state. Thus, by choosing $\sigma_r = \sqrt{2} - 1$, we obtain

$$
||\mathcal{S}_1 - \mathcal{S}_2|| \leq 2\sqrt{2} - 2.
$$

6. Consider a system described by the following transfer function matrix

$$
G(s) = \begin{bmatrix} \frac{1}{1+s} & \frac{-1}{s+1} \\ \frac{2s}{s+2} & \frac{2s}{s+2} \end{bmatrix}
$$

Suggest a suitable transformation for doing decoupled control and provide the overall structure of the controller. (You do not have to design any specific controller for the decoupled systems) (2 p)

Solution

We notice that the first output is driven by the difference of the control signal, the second output is driven by the sum, we can thus make a coordinate change in the output of our controllers exploiting this information.

$$
\begin{aligned}
u &= F\hat{u}, \\
\hat{U}_1 &= C_1 Y_1, \\
\hat{U}_2 &= C_2 Y_2 \\
F &= \begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix} \\
\implies Y &= GF\hat{U} = \begin{bmatrix} \frac{1}{1+s} & 0 \\ 0 & \frac{2s}{s+2} \end{bmatrix} \hat{U}
\end{aligned}
$$

We thus see that this choice of F completely decouples our system.

7.

a. Design an internal model controller for the system

$$
P(s) = \begin{pmatrix} \dfrac{s-1}{s+2} & 0 \\ 0 & \dfrac{1}{s+4} \end{pmatrix}.
$$

If you have the possibility to decide the crossover frequency ω_c for the open loop of any of the subsystems, choose your design parameters such that $\omega_c = 2$ rad/s. (3 p)

b. Internal model controllers can also be designed by solving a convex optimization problem of the form

$$
\min_Q ||T_1 + T_2 Q||.
$$

Here T_1 and T_2 can be chosen so that

$$
T_1 + T_2 Q = \begin{pmatrix} W_S S \\ W_T T \end{pmatrix},
$$

where $S = (I + PC)^{-1}$ is the sensitivity function and $T = PC(I + PC)^{-1}$ is the complementary sensitivity function and W_S as well as W_T are weighting functions. After finding the optimal Q, the controller can be obtained as $C = (I - QP)^{-1}Q$. Find expressions for T_1 and T_2 in terms of a general process P as well as general weighting functions W_S and W_T.

Hint: Use $S + T = I$! (2 p)

Solution

a. By following the recommendations in the book for how to handle non-minimum phase zeros and how to make Q proper we get

$$Q(s) = \begin{pmatrix} \frac{s+2}{s+1} & 0 \\ 0 & \frac{s+4}{\lambda s+1} \end{pmatrix}.$$

The corresponding controller is given by

$$C(s) = (I - Q(s)P(s))^{-1}Q(s) = \begin{pmatrix} \frac{s+1-(s-1)}{s+1} & 0 \\ 0 & \frac{\lambda s+1-1}{\lambda s+1} \end{pmatrix}^{-1} Q(s) =$$

$$= \begin{pmatrix} \frac{s+1}{2} & 0 \\ 0 & \frac{\lambda s+1}{\lambda s} \end{pmatrix} \begin{pmatrix} \frac{s+2}{s+1} & 0 \\ 0 & \frac{s+4}{\lambda s+1} \end{pmatrix} = \begin{pmatrix} \frac{s+2}{2} & 0 \\ 0 & \frac{s+4}{\lambda s} \end{pmatrix}.$$

The open-loop transfer function is

$$P(s)C(s) = \begin{pmatrix} \frac{s-1}{s+2} & 0 \\ 0 & \frac{1}{s+4} \end{pmatrix} \begin{pmatrix} \frac{s+2}{2} & 0 \\ 0 & \frac{s+4}{\lambda s} \end{pmatrix} = \begin{pmatrix} \frac{s-1}{2} & 0 \\ 0 & \frac{1}{\lambda s} \end{pmatrix}.$$

If we choose the design parameter $\lambda = 0.5$, the subsystem (2,2) will get the crossover frequency $\omega_c = 2$ rad/s.

b. We first solve the controller equation for Q, that is

$$\begin{aligned} & C = (I - QP)^{-1}Q \\ \iff & (I - QP)C = Q \\ \iff & C = Q + QPC \\ \iff & C(I + PC)^{-1} = Q. \end{aligned}$$

Thus

$$\begin{aligned} T_1 + T_2 Q &= \begin{pmatrix} W_S S \\ W_T T \end{pmatrix} = \begin{pmatrix} W_S(I - T) \\ W_T T \end{pmatrix} = \begin{pmatrix} W_S\left(I - PC(I+PC)^{-1}\right) \\ W_T\left(PC(I+PC)^{-1}\right) \end{pmatrix} \\ &= \begin{pmatrix} W_S(I - PQ) \\ W_T PQ \end{pmatrix} = \underbrace{\begin{pmatrix} W_S \\ 0 \end{pmatrix}}_{T_1} + \underbrace{\begin{pmatrix} -W_S P \\ W_T P \end{pmatrix}}_{T_2} Q. \end{aligned}$$