1.

a. We are not asked to determine the multiplicity of poles, so it is sufficient to find the poles of the individual matrix elements. Hence we find the poles $s = -1$, $s = -5$ and $s = -2.5$.

To find the zeros, we compute the maximal subdeterminant

$$\det G(s) = \begin{vmatrix} \frac{2.4}{s+1} & -\frac{16}{s+5} \\ \frac{1.8}{s+1} & -\frac{12}{2s+5} \end{vmatrix}$$
$$= -\frac{28.8}{(s+1)(2s+5)} + \frac{28.8}{(s+1)(s+5)}$$
$$= \frac{28.8\ s}{(s+1)(2s+5)(s+5)}$$

and find that the only zero is located at $s = 0$.

b. The matrices are orthogonal as they should, i.e. $UU^T = VV^T = I$. Hence Σ can be computed as

$$\Sigma = U^T G(0) V = \begin{pmatrix} -0.8 & -0.6 \\ -0.6 & 0.8 \end{pmatrix} \begin{pmatrix} 2.4 & -16/5 \\ 1.8 & -12/5 \end{pmatrix} \begin{pmatrix} -0.6 & -0.8 \\ 0.8 & -0.6 \end{pmatrix}$$
$$= \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}$$

so the singular values are 5 and 0.

c. The singular vector corresponding to the singular value 5 is the first column of the matrix V. Hence the input vector with strongest amplification is

$$u = \begin{pmatrix} -0.6 \\ 0.8 \end{pmatrix}$$

This is easily verified by computing

$$y = G(0)u = \begin{pmatrix} 2.4 & -16/5 \\ 1.8 & -12/5 \end{pmatrix} \begin{pmatrix} -0.6 \\ 0.8 \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}$$
$$|y| = \sqrt{(-4)^2 + (-3)^2} = 5$$

Similarly, the minimal output norm zero is obtained when u equals the second column of V, i.e.

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2.4 & -16/5 \\ 1.8 & -12/5 \end{pmatrix} \begin{pmatrix} -0.8 \\ -0.6 \end{pmatrix}$$

d. From lecture 6, the transfer matrix $\sum_{i=1}^{n} \frac{C_i B_i}{s-p_i} + D$ has the realization

$$\dot{x}(t) = \begin{bmatrix} p_1 I & & 0 \\ & \ddots & \\ 0 & & p_n I \end{bmatrix} x(t) + \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} u(t)$$

$$y(t) = [\, C_1 \quad \dots \quad C_n \,]\, x(t) + Du(t)$$

Hence

$$G(s) = \begin{pmatrix} \frac{2.4}{s+1} & -\frac{16}{s+5} \\ \frac{1.8}{s+1} & -\frac{12}{2s+5} \end{pmatrix} = \frac{1}{s+1} \underbrace{\begin{pmatrix} 2.4 \\ 1.8 \end{pmatrix}}_{C_1} \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{B_1}$$

$$+ \frac{1}{s+5} \underbrace{\begin{pmatrix} -16 \\ 0 \end{pmatrix}}_{C_2} \underbrace{\begin{pmatrix} 0 & 1 \end{pmatrix}}_{B_2} + \frac{1}{s+2.5} \underbrace{\begin{pmatrix} 0 \\ -6 \end{pmatrix}}_{C_3} \underbrace{\begin{pmatrix} 0 & 1 \end{pmatrix}}_{B_3}$$

with $p_1 = -1$, $p_2 = -5$ and $p_3 = -2.5$ this gives the state space realization

$$\dot{x} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -2.5 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 2.4 & -16 & 0 \\ 1.8 & 0 & -6 \end{pmatrix} x$$

2.

a. We can not construct a dynamic decoupling $W(s)$ that gives $G(s)W(s) = I$, since we can not take the inverse of the system matrix due to the time delay. Hence, we must go for the second best, decoupling at steady state. This is easily achieved by using the inverse of the static gain, i.e.,

$$W(s) = G(0)^{-1} = \frac{1}{5} \begin{pmatrix} -1 & 2 \\ 3 & -1 \end{pmatrix}$$

b.

$$\lim_{s \to 0} sC(s) = I$$

$$\lim_{s \to 0} [sI + G(s)W(s)sC(s)]^{-1} = \left[G(0)W(0) \lim_{s \to 0} sC(s) \right]^{-1} = I$$

$$S(0) = \lim_{s \to 0} \left(s[sI + G(s)W(s)sC(s)]^{-1} \right) = 0$$

$$T(0) = I - S(0) = I$$

The integrators in $C(s)$ give infinitely high gain at low frequencies, hence the denominator of $S(s)$ grows near $s = 0$ to make $S(0) = 0$. The argument fails if $G(0)W(0) \lim_{s \to 0} sC(s)$ is singular.

3.

a. The controllability Gramian is the solution to the following equation:

$$AS + SA^T + BB^T = 0$$

$$\begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}+\begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}\begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix}^T+\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}^T=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

where the fact that the controllability gramian is symmetric has been used. Performing the matrix multiplications results in the following linear equation system:

$$\begin{cases} -6s_1 + 2s_2 = 0 \\ -4s_2 + s_1 + s_3 = 0 \\ 2s_2 - 2s_3 + 1 = 0 \end{cases}$$

which has the solution

$$\begin{cases} s_1 = 1/16 \\ s_2 = 3/16 \\ s_3 = 11/16 \end{cases}$$

and the controllability Gramian is given by

$$S = \frac{1}{16}\begin{pmatrix} 1 & 3 \\ 3 & 11 \end{pmatrix}$$

The eigenvalues of S are the solutions of the characteristic equation

$$\det(\lambda I - S) = \begin{vmatrix} \lambda - 1/16 & -3/16 \\ -3/16 & \lambda - 11/16 \end{vmatrix} = (\lambda - \frac{1}{16})(\lambda - \frac{11}{16}) - \frac{9}{16^2} = \lambda^2 - \frac{3}{4}\lambda + \frac{2}{16^2} = 0$$

which has the solutions

$$\lambda = \frac{3}{8} \pm \sqrt{\left(\frac{3}{8}\right) - \frac{2}{16^2}} = \frac{3}{8} \pm \frac{\sqrt{34}}{16}$$

with numerical values $\lambda_1 \approx 0.74$ and $\lambda_2 \approx 0.011$. The conclusion to draw from this is that there will be one direction that is much more costly to control in, namely the eigenvector corresponding to λ_2, as $\lambda_1 \approx 70 \cdot \lambda_2$. (as the cost to control the system from $x(0) = 0$ to $x(t_1) = x_1$ is bounded from below by $\int_0^{t_1} u^2 dt \geq x_1^T S^{-1} x_1$)

b. As the system now is balanced, the controllability Gramian will be diagonal and the Hankel singular values will be the diagonal elements of the controllability Gramian. First partition the system matrices as

$$A_{bal} = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_3 \end{pmatrix} \quad B_{bal} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \quad C_{bal} = \begin{pmatrix} C_1 & C_2 \end{pmatrix}$$

The controllability Gramian is calculated from the following equation

$$A_{bal}S + SA_{bal}^T + B_{bal}B_{bal}^T = 0$$

Using the fact that S will be diagonal results in

$$\begin{pmatrix} A_1 & A_2 \\ A_2 & A_3 \end{pmatrix} \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} + \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ A_2 & A_3 \end{pmatrix}^T + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

results in the following linear equation system

$$\begin{cases} 2A_1s_1 + B_1^2 = 0 \\ A_2(s_1 + s_2) + B_1B_2 = 0 \\ 2A_3s_2 + B_2^2 = 0 \end{cases}$$

which has the solution

$$\begin{cases} s_1 = \dfrac{-B_1^2}{2A_1} = \dfrac{59 + 10\sqrt{34}}{8\left(14 + \sqrt{34}\right)} \approx 0.74 \\ s_2 = \dfrac{-B_2^2}{2A_3} = \dfrac{9}{8\left(54 + 9\sqrt{34}\right)} \approx 0.011 \end{cases}$$

This solution fulfills all of the three equations in the equation system.

c. Partition the system as

$$\begin{cases} \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u \\ y = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \end{cases}$$

The reduced system is then given by (reduction of the second state)

$$\begin{cases} \dot{z}_1 = \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right) z_1 + \left(B_1 - A_{12}A_{22}^{-1}B_2\right) u \\ y_r = \left(C_1 - C_2A_{22}^{-1}A_{21}\right) z_1 - C_2A_{22}^{-1}B_2u \end{cases}$$

The reduced system is hence given by

$$\begin{cases} \dot{z}_1 = -\dfrac{2006 + 340\sqrt{34}}{3366 + 576\sqrt{34}} z_1 + \dfrac{187 + 32\sqrt{34}}{\left(18 + 3\sqrt{34}\right)\sqrt{68 + 10\sqrt{34}}} u \\ y_r = \dfrac{187 + 32\sqrt{34}}{\left(18 + 3\sqrt{34}\right)\sqrt{68 + 10\sqrt{34}}} z_1 + \dfrac{34 + 5\sqrt{34}}{1496 + 256\sqrt{34}} u \end{cases}$$

and with numerical values this becomes

$$\begin{cases} \dot{z}_1 = -0.59z_1 + 0.94u \\ y_r = 0.94z_1 + 0.021u \end{cases}$$

4. The transfer function from n to the system output y is $S(s) = \frac{s(s+1)}{s^2+s+1}$. This function goes to zero for small frequencies, has a small resonance peak at $\omega = 1$ and tends to 1 for high frequencies. By multiplying $S(s)$ with the different noise filters, we can get an idea what the frequency content of the output signal will be.

The filter H_3 is resonant with a gain peak at $\omega_3 = 1$, while H_1 and H_2 are both low-pass filters with cut-off frequencies $\omega_1 = 0.1$ and $\omega_2 = 10$ respectively.

The resonant filter has its gain peak at the same frequency as $S(s)$, which means that the output will be dominated by this frequency. The resonance frequency $\omega_3 = 1$ corresponds to a period time of $T = 2 * \pi \approx 6.3$, and we see that realization B looks almost sinusoidal with this period time.

Of the remaining realizations, A varies faster (it has more high-frequency content) and C varies slower (less high-frequency content). Thus H_1 corresponds to C and H_2 corresponds to A.

5.

a. What your boss says is essentially that he wants $y(t) = r(t)$ for all t. This is equivalent to $Y(s) = R(s)$, so we should make the transfer function $G_{yr} = \frac{FCP}{1+CP}$ equal to 1. This would require

$$F = \frac{1+CP}{CP} = \frac{1+\frac{10s+1}{s^3+6s^2+9s}}{\frac{10s+1}{s^3+6s^2+9s}} = \frac{s^3+6s^2+19s+1}{10s+1} \tag{1}$$

which is non-proper, hence impossible to realize.

b. The equation (1) gives $F(0) = 1$ and

$$|F(100i)| = 999.9$$

Hence, exact reference following at high frequencies requires high gain.

c. Once again, the transfer function from r to y is $G_{yr} = \frac{FCP}{1+CP}$. Hence the error amplitude A is given by

$$\begin{aligned} A^2 = |G_{yr}(i) - 1|^2 &= \left| F(i)\frac{C(i)P(i)}{1+C(i)P(i)} - 1 \right|^2 \\ &= \left| \left(f_0 + \frac{f_1}{i+1} + \frac{f_2}{i+10} \right) \frac{10i+1}{i^3+6i^2+19i+1} - 1 \right|^2 \end{aligned}$$

This is a quadratic expression in f_0, f_1 and f_2. This is the objective function to minimize.

Zero stationary error $y-r$ when $r(t)$ is constant means that $F(0) = 1$. This means that

$$f_0 + f_1 + \frac{f_2}{10} = 1$$

This is the constraint to take into account when minimizing the objective function previously calculated.

6.

a. The Riccati equation for the state feedback part is ($y = Cx$)

$$0 = C^T Q_1 C + A^T S + SA - (SB + Q_{12})Q_2^{-1}(SB + Q_{12})^T$$

with $A = -1$, $B = 1$, $C = 1$, $Q_1 = 2$, $Q_{12} = 0$, $Q_2 = 1$ becomes

$$0 = 2 - 2S - S^2 = 3 - (S+1)^2$$

which gives the positive semidefinite solution

$$S = -1 + \sqrt{3}$$

and $L = S$. Hence the optimal state feedback is

$$u = (1 - \sqrt{3})x$$

For the Kalman filter we have the Riccati equation

$$0 = R_1 + AP + PA^T - (PC^T + R_{12})R_2^{-1}(PC^T + R_{12})^T$$

and $C = R_1 = R_2 = 1$, $R_{12} = 0$ gives

$$0 = 1 - 2P - P^2 = 2 - (P+1)^2$$

with the solution $P = -1+\sqrt{2}$ and Kalman filter gain $K = PC^T = -1+\sqrt{2}$. Thus, the controller is

$$\begin{aligned} \dot{\hat{x}}(t) &= -\hat{x}(t) + u(t) + (-1 + \sqrt{2})(y(t) - C\hat{x}(t)) \\ u(t) &= (1 - \sqrt{3})\hat{x}(t) \end{aligned}$$

b. The closed loop system poles are determined by $A - BL$ and $A - KC$ which thus are $-\sqrt{3}$ and $-\sqrt{2}$, respectively.

c. Minimization of

$$\mathbf{E}\left(y(t)^2 + \psi u(t)^2\right) = \psi \mathbf{E}\left(\frac{1}{\psi}y(t)^2 + u(t)^2\right)$$

is the same as minimization of

$$\mathbf{E}\left(\frac{1}{\psi}y(t)^2 + u(t)^2\right)$$

Hence, using $\psi = \dfrac{1}{2}$ will give the same control law as before.