Solutions to Exam in FRTN10 Multivariable Control 2018-10-27

1 a. From the block diagram we can easily find that

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{-2}{s+1} \\ \frac{3}{s+1} & \frac{3(s-1)}{(s+1)(s+3)} \end{bmatrix}$$

b. One can immediately see from the block diagram that the system has two poles, one in -1 and one in -3.

Alternatively, one can calculate all minors and find the least common denominator. The relevant 1st order subdeterminants are the four non-zero elements

$$\frac{1}{s+1}, \quad \frac{-2}{s+1}, \quad \frac{1}{s+1}, \quad \frac{3(s-3)}{(s+1)(s+3)}$$

and the 2nd order subdeterminant is

$$\frac{9(s+1)}{(s+1)^2(s+3)} = \frac{9}{(s+1)(s+3)}$$

The least common denominator of all subdeterminants is

$$p(s) = (s+1)(s+3).$$

The system therefore has two poles: one in -1 and one in -3.

The maximal minor is the 2nd order subdeterminant $\frac{9}{(s+1)(s+3)}$; the zero polynomial is just a constant, and therefore the system has no zeros.

c. The transfer function from u_1 to y_2 is $G_{21}(s) = \frac{3}{s+1}$. To find the variance of y_2, we can either directly calculate

$$V(y_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(i\omega)G(-i\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d\omega}{\omega^2+1} = \frac{\text{atan}(\infty)-\text{atan}(-\infty)}{2\pi} = \frac{9}{2}$$

or introduce the state-space realization $(A,B,C,D) = (-1,3,1,0)$ and solve the Lyapunov equation for $\Pi = V(y_2)$ as

$$A\Pi + \Pi A^T + BB^T = 0 \Leftrightarrow -2\Pi + 9 = 0 \Leftrightarrow \Pi = \frac{9}{2}$$

2. From the dimensions given (3 poles/states, 1 input, 2 outputs), the A matrix should be 3×3, the B matrix 3×1 and the C matrix 2×3. The easiest option is to choose a system in diagonal form with some stable eigenvalues (LHP poles) on the diagonal, for example

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} x$$

With a zero in the top B element, we immediately see that the first state is not controllable. Alternatively we can compute the controllability matrix $\mathscr{C} = [B \quad AB \quad A^2B]$ and note that the first row is all zeros $\Rightarrow$ not full rank.

With y_1, we directly measure x_1. With $y_2 = x_2 + x_3$, we can observe both states since they have different eigenvalues. Alternatively we can compute the observability matrix $\mathscr{O} = [C^T \quad A^T C^T \quad (A^T)^2 C^T]^T$ and see that it has full rank. (Any other C matrix that includes all three states would also work.)

The transfer function has the form $\frac{N_1}{s+1} + \frac{N_2}{s+2} + \frac{N_3}{s+3}$, which is even strictly proper (if $D = 0$). Alternatively, we can directly calculate $G(s)$ and verify that it is proper.

3 a. From the block diagram, we have

$$
\begin{aligned}
-QPw + R(F(w-v) - GPw) = v \quad &=> \quad (I + RF)v = (RF - RGP - QP)w \\
&=> \quad v = (I + RF)^{-1}(RF - RGP - QP)w \\
&=> \quad H = (I + RF)^{-1}(RF - RGP - QP)
\end{aligned}
$$

b. To be able to conclude stability of the closed-loop system with the Small Gain Theorem, in addition to having a stable Δ_i, we should have

$$\|\Delta_i\| < \frac{1}{\|H\|_\infty} = 1/4$$

Δ_2 is not stable, so we can not use Small Gain Theorem for it. The other blocks have the maximum gains

$$\|\Delta_1\| = 1/8, \qquad \|\Delta_3\| = 1/2, \qquad \|\Delta_4\| = 1/5$$

We can hence guarantee stability of the control loop with Δ_1 and Δ_4.

4 a. Yes. This can be seen in many ways, for instance in that $P/(1+PC)$ approaches zero for low frequencies, indicating that a constant load disturbance can be rejected. It can also be seen in the difference at low frequencies between $C/(1+PC)$ and $1/(1+PC)$.

b. Yes. The maximum sensitivity is smaller than 2, indicating a gain margin of at least 2. The maximum complementary sensitivity is even smaller, implying robust stability.

c. In the curve for $C/(1+PC)$ we read out the maximum gain ≈ 9.

d. No. The time delay imposes an upper limit of the achievable bandwidth of about $1.6/L = 16$ rad/s, and the current bandwidth is only somewhat lower than this.

e. The curve for $PCF/(1+PC)$ shows that the step response will be smooth and well-damped, with a bandwidth of about 10 rad/s. The process time delay will also show up in the step response. The static gain is 1.

5 a. Since G is triangular, we immediately know that $RGA = I$. Alternatively, we can calculate the RGA according to

$$G(0) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}, \qquad G(0)^{-1} = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

$$RGA(G(0)) = G(0) \mathbin{.*} G(0)^{-T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From this, a suitable pairing is input 1 with output 1, input 2 with output 2 and input 3 with output 3.

This is indeed the only reasonable choice. From the transfer matrix we see that y_3 is only affected by u_3, so that pairing is given. If u_3 is used for y_3 then u_2 has to be used for y_2 since it's unaffected by u_1. This leaves u_1 for y_1. All other decentralized schemes would have an output that couldn't be controlled.

b. For its input pairing, the last output has the transfer function $P = \frac{1}{s-1}$. We can write out the requirement for first order roll-off as

$$|T(i\omega)| \leq \left|\frac{k}{i\omega}\right| \Leftrightarrow \left|\frac{i\omega}{k}T(i\omega)\right| \leq 1 \Leftrightarrow \left\|\frac{i\omega}{k}T(i\omega)\right\|_\infty \leq 1$$

However, we know that the rightmost condition can only be satisfied if $|\frac{p}{k}| \leq 1$ for all unstable poles p of the process. We have an unstable pole in 1, which gives

$$\frac{1}{k} \leq 1$$

The conclusion is that k cannot be smaller than 1.

6 a. From the block diagram (setting $C = 0$) we get

$$P = \begin{bmatrix} 0 & P_0 & 0 & P_0 \\ 1 & P_0 & 0 & P_0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

with

$$P_{z\omega} = \begin{bmatrix} 0 & P_0 & 0 \end{bmatrix} \quad P_{zu} = \begin{bmatrix} P_0 \end{bmatrix}$$
$$P_{y\omega} = \begin{bmatrix} 1 & P_0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P_{yu} = \begin{bmatrix} P_0 \\ 0 \end{bmatrix}$$

b. We need to add the error to the output of our model, i.e.

$$z = \begin{bmatrix} x \\ x - r \end{bmatrix}$$

resulting in

$$P = \begin{bmatrix} 0 & P_0 & 0 & P_0 \\ 0 & P_0 & -1 & P_0 \\ 1 & P_0 & 0 & P_0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

c.

$$G_{cl} = P_{z\omega} + P_{zu} Q P_{y\omega}$$

The Youla parametrization is preferred since the closed-loop transfer function then is linear (actually affine) in the design variable. Without this the resulting optimization problem is much harder to solve.

7 a. Noting that A and Q_{12} are zero matrices and Q_2 is the identity results in the following simplification of the Riccati equation:

$$Q_1 - SBB^T S = 0$$

Setting

$$S = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$$

and noting that $BB^T = I$ results in the following system of equations:

$$\begin{aligned} 3 &= a^2 + c^2 \\ 1 &= b^2 + c^2 \\ 0 &= (a+b)c \end{aligned}$$

From the last equation we can conclude that $c = 0$, since otherwise either a or b needs to be negative and a positive definite matrix can't have negative diagonal elements. This results in

$$S = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix}$$

and the feedback law is given by

$$u = -(SB)^T x = -\begin{bmatrix} 0 & 1 \\ \sqrt{3} & 0 \end{bmatrix} x$$

b. The system description with noise is

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} u + w_1 \\ y &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + w_2 \end{aligned}$$

where w_1 and w_2 are two-dimensional white noise processes with intensities $R_1 = R_2 = I$. Since A and R_{12} are zero and R_1, R_2 and C are the identity we get the Riccati equation

$$I = PP$$

Setting

$$P = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$$

results in

$$\begin{aligned} 1 &= a^2 + c^2 \\ 1 &= b^2 + c^2 \\ 0 &= (a+b)c \end{aligned}$$

and with similar reasoning as in the previous question we get

$$P = I$$

and the Kalman gain becomes

$$K = PC^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

And the Kalman filter is given by

$$\dot{\hat{x}} = -\hat{x} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} u + y$$

c. The separation principle for LQG control says that the feedback law we calculated in the fully observed case still is optimal in the uncertain case.

d. A reasonable suggestion for the source of the high-frequency components would be high-frequency measurement noise. In order for the Kalman filter to take greater consideration of this noise we need to model it. Therefore, scaling R_2 to make it larger (or scaling R_1 to make it smaller) would be a sensible first step in the tuning process.