

**Solutions to Exam in FRTN10 Multivariable Control 2018-04-06**

**1 a.** The pole polynomial is given by the LCD of the minors of  $G$ ,

$$\frac{1}{s+1}, \quad \frac{-1}{s+1}, \quad \frac{2s}{s+1}, \quad \frac{2s}{s+2}, \quad \frac{2s(2s+3)}{(s+1)^2(s+2)}$$

which is  $(s+1)^2(s+2)$ . The poles are hence  $-1$  (mult. 2) and  $-2$  (mult. 1).  
The zero polynomial is given by the GCD of the maximal minor of  $G$  normalized with the pole polynomial, which is  $\frac{2s(2s+3)}{(s+1)^2(s+2)}$ . The zeros are hence 0 and  $-1.5$  (both mult. 1).

**b.** We rewrite the transfer matrix as (for instance)

$$G(s) = \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix}}{s+1} + \frac{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \end{bmatrix}}{s+1} + \frac{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & -4 \end{bmatrix}}{s+2} + \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}$$

from which we get the state-space realization

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 & -1 \\ -2 & 0 \\ 0 & -4 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} u \end{aligned}$$

**c.** When the input frequency goes to infinity,

$$\lim_{\omega \rightarrow \infty} G(i\omega) = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} = D$$

The given SVD  $D = U\Sigma V^*$  shows that the maximal singular value is  $2\sqrt{2}$ , and this is by definition the maximum gain. The input direction is given by the corresponding column vector of  $V$ ,  $v = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$ .

**d.** The RGA in stationarity is given by  $G(0) \cdot G(0)^{-T}$ , which is not possible to compute since  $G(0) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$  is singular. The fundamental problem is the transmission zero in 0, which makes it impossible to control the system in stationarity.

**e.** The transformation corresponds to the coordinate change  $u_1 = \tilde{u}_1 + \tilde{u}_2$ ,  $u_2 = \tilde{u}_2$ , or, equivalently,  $W_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ . The transformed process becomes

$$PW_1 = \begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{2s}{s+1} & \frac{2s(2s+3)}{(s+1)(s+2)} \end{bmatrix}$$

which shows that  $y_1$  is only affected by  $\tilde{u}_1$ .

2. The plant is non-minimum phase and has the poles  $s = -2$  (slowest) and  $s = -5$ . We go for the approach of mirroring the unstable zero before inverting the system. This results in

$$Q(s) = \frac{s^2 + 7s + 10}{(4 + 2s)(\lambda s + 1)^n}$$

This choice of  $Q$  gives the closed loop static gain  $P(0)Q(0) = 1$ . We choose  $n = 1$  to get a proper  $Q$  and a proper controller. Matching the slowest pole, we choose  $\lambda = 0.5$ , which results in

$$Q(s) = \frac{s + 5}{s + 2}$$

With this particular choice of  $Q$  the controller becomes

$$C(s) = \frac{Q(s)}{1 - Q(s)P(s)} = \frac{s^2 + 7s + 10}{s(s + 6)}$$

The degree of the numerator is not higher than that of the denominator, so the controller is indeed proper.

- 3 a.  $v$  is a (column) vector of length 2,  $y$  is a scalar, and  $F(s)$  is a  $2 \times 1$  system (1 input and 2 outputs).

- b. We can use the Small Gain Theorem to guarantee stability if  $F$  is stable and  $\|F\| < 1/\|G\|$ . From the plot we read out  $\|G\| \approx 2.12$ , which gives the condition  $\|F\| < 0.47$ .

- c. We can take for instance  $F(s) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ , which yields the closed loop system

$$\frac{G(s)}{1 + G(s)F(s)} = \begin{bmatrix} \frac{2s(s + 3)}{(s + 1)(s - 1)} & \frac{4}{s - 1} \end{bmatrix}$$

with an unstable pole in  $s = 1$ .

4. We see that  $P_{pF}$  has a right-half-plane zero in  $z = 4$ , and a right-half-plane pole in  $p = 3$ . In lecture 7, we find the following equation that gives a lower bound of the maximum value of the sensitivity function

$$\sup_{\omega} |S(i\omega)| \geq \frac{p + z}{|p - z|} = 7$$

When the sensitivity function is greater than 2-3, the robustness of the controller is too poor to be practically useful. This shows that it is impossible to achieve good control performance.

- 5 a. From lecture slide we get the following Riccati equation for LQ problem and Kalman gain

$$\begin{aligned} K &= (PC^T + NR_{12})R_2^{-1} & L &= Q_2^{-1}(SB + Q_{12})^T \\ 0 &= Q_1 + A^T S + SA - (SB + Q_{12})Q_2^{-1}(SB + Q_{12})^T \\ 0 &= NR_1 N^T + AP + PA^T - (PC^T + NR_{12})R_2^{-1}(PC^T + NR_{12})^T \end{aligned}$$

where in this problem we have

$$A = 1 \quad B = 1 \quad M = 1 \quad N = 1 \quad C = 1 \quad Q_{12} = 0 \quad R_{12} = 0$$

From the LQ Riccati equation one can write

$$\begin{aligned} S + S + Q_1 - S^2/Q_2 = 0 & \Rightarrow S = Q_2(1 + \sqrt{1 + Q_1/Q_2}) \\ & \Rightarrow L = Q_2^{-1}S = 1 + \sqrt{1 + q} \end{aligned}$$

Similarly for Kalman filter we have

$$\begin{aligned} P + P + R_1 - P^2/R_2 = 0 & \Rightarrow P = R_2(1 + \sqrt{1 + R_1/R_2}) \\ & \Rightarrow K = PR_2^{-1} = 1 + \sqrt{1 + r} \end{aligned}$$

Note that  $P$  and  $S$  cannot take negative values. Since controllers have the form  $u = -L\hat{x}$  with  $\frac{d}{dt}\hat{x} = \hat{x} + u + K[y - \hat{x}]$ , and  $L$  is a function of  $q$  and  $K$  is a function of  $r$ , the controller is only a function of  $r$  and  $q$ .

- b.** From the lecture slides, the closed-loop dynamics is

$$\frac{d}{dt} \begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix} = \begin{bmatrix} A - BL & BL \\ 0 & A - KC \end{bmatrix} \begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix} + \begin{bmatrix} Nv_1(t) \\ Nv_1(t) - Kv_2(t) \end{bmatrix}$$

in this problem, both the plant and the observer have first-order dynamics. We know that poles of the closed-loop system are the eigenvalues of  $A - BL$  and  $A - KC$ . Here we have

$$\begin{aligned} A - KC &= 1 - (1 + \sqrt{1 + r}) = -\sqrt{1 + r} \\ A - BL &= 1 - (1 + \sqrt{1 + q}) = -\sqrt{1 + q} \end{aligned}$$

- c.** Putting larger penalty on the state variable than the control signal means  $Q_1 \gg Q_2$ , as a result we have  $q \gg 1$ , then the process pole goes farther to the left (to large negative values), that is the process response is faster (goes faster to the origin). In the other words, since we get a large  $q$ ,  $L$  would be larger and it means that we get a larger control signal (In another words, since  $Q_2$  is small, the control signal is allowed to have larger values, so it results in a faster response). On the other hand, if  $Q_1 \ll Q_2$  then  $q \ll 1$ , we are not allowed to have large control signals (since  $q \ll 1$ ,  $L$  would be small) and the process pole approaches  $-1$ , that is the process response would be slower compared to the other case.
- d.** If we put more trusts on measurements we get  $R_1 \gg R_2$ , hence  $r \gg 1$ , then the observer pole goes farther to the left, that is we get a faster convergence rate for the filter. On the other hand, if we trust the process more than the measurements, we have  $R_1 \ll R_2$  then  $r \ll 1$ , so the observer pole approaches  $-1$ , that is we get slower convergence rate.
- 6 a.** In the balanced realization the observability Gramian and the controllability Gramian are equal so  $S_\xi = O_\xi$ .

b. Consider a balanced realization

$$\begin{bmatrix} \dot{\hat{\xi}}_1 \\ \dot{\hat{\xi}}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \quad , \quad \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

$$y = [C_1 \quad C_2] \begin{bmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \end{bmatrix} + Du$$

where  $\Sigma_2$  corresponds to the small Hankel singular values. Replacing the second state equation by  $\dot{\hat{\xi}}_2 = 0$ , gives the reduced system below

$$\begin{aligned} \dot{\hat{\xi}}_1 &= (A_{11} - A_{12}A_{22}^{-1}A_{21})\hat{\xi}_1 + (B_1 - A_{12}A_{22}^{-1}B_2)u \\ y_{red} &= (C_1 - C_2A_{22}^{-1}A_{21})\hat{\xi}_1 + (D - C_2A_{22}^{-1}B_2)u \end{aligned}$$

after carrying out some calculations, we get

$$\begin{aligned} \dot{\hat{\xi}}_1 &= -0.448 \hat{\xi}_1 + [-0.873 \quad -2.45] u \\ y_{red} &= \begin{bmatrix} -0.350 \\ -2.58 \end{bmatrix} \hat{\xi}_1 + \begin{bmatrix} 0.317 & 0.330 \\ -0.0354 & -0.897 \end{bmatrix} u \end{aligned}$$

c. No, since  $\xi_2$  and  $\xi_3$  has almost the same influence which is small in comparison to the influence of the first state it should not make a big difference if you remove both.