**Solutions to Exam in FRTN10 Multivariable Control 2017-10-27**

**1 a.** The minors are easily found as

$$\frac{1}{(s+2)(s+3)}, \quad \frac{1}{(s+3)(s+4)}, \quad \frac{1}{(s+2)(s+4)}$$

Rewrite them with the least common denominator:

$$\frac{s+4}{(s+2)(s+3)(s+4)}, \quad \frac{s+2}{(s+2)(s+3)(s+4)}, \quad \frac{s+3}{(s+2)(s+3)(s+4)}$$

It's clear from this that the poles are given by $-2$, $-3$, and $-4$, all with multiplicity 1.

The zeros are given by the nominators' greatest common divisor, which here is 1. The system therefore has no zeros.

**b.** Do a partial fraction decomposition of the entries of $G$:

$$G(s) = \begin{bmatrix} \frac{1}{s+2} - \frac{1}{s+3} & \frac{1}{s+3} - \frac{1}{s+4} & \frac{\frac{1}{2}}{s+2} - \frac{\frac{1}{2}}{s+4} \end{bmatrix}$$

Separate the terms according to the poles:

$$G(s) = \frac{\begin{bmatrix} 1 & 0 & \frac{1}{2} \end{bmatrix}}{s+2} + \frac{\begin{bmatrix} -1 & 1 & 0 \end{bmatrix}}{s+3} + \frac{\begin{bmatrix} 0 & -1 & -\frac{1}{2} \end{bmatrix}}{s+4}$$

The matrices are trivially subdivided as (for instance)

$$G(s) = \frac{[1]\begin{bmatrix} 1 & 0 & \frac{1}{2} \end{bmatrix}}{s+2} + \frac{[1]\begin{bmatrix} -1 & 1 & 0 \end{bmatrix}}{s+3} + \frac{[1]\begin{bmatrix} 0 & -1 & -\frac{1}{2} \end{bmatrix}}{s+4}$$

A minimal state-space realization is now given by (for instance)

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ -1 & 1 & 0 \\ 0 & -1 & -\frac{1}{2} \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x(t)$$

**2.** $P_3$ has an unstable zero in 0.2, which means that the closed-loop bandwidth cannot be greater than 0.2 rad/s. This corresponds to step response D, which is both very slow and has inverse initial response. It also corresponds to sensitivity plot I, which has a bandwidth of approx. 0.2 rad/s.

$P_4$ has an unstable pole in 10, which means that the closed-loop bandwidth cannot be smaller than 10 rad/s. The corresponds to step response B, which has a time constant in the order of 0.1. It also corresponds to sensitivity plot III, which has a bandwidth higher than 10 rad/s.

$P_2$ has a time delay of 1 second, which limits the closed-loop bandwidth to approx. 1 rad/s. The time delay is clearly visible in step response A. The step response has an overshoot, which corresponds to the resonance peak in $T$ visible in sensitivity plot IV.

$P_1$ does not have any fundamental limitations, so it could have any closed-loop bandwidth and characteristics, depending on the controller. The only remaining diagrams are C and II, that correspond well together, with a time constant of approx. 1 s and a bandwidth of approx. 1 rad/s.

**3.** We want the system to behave like a diagonal system $y_d = \widetilde{P}u_d$ in stationarity with the controller $u_d = -C_d y_d$. This means we want to find some constant coordinate transformations, $W_1$ and $W_2$, such that $\widetilde{P}(0) = W_2 P(0) W_1$.

For simplicity we choose $W_1 = I$ and $W_2$ is chosen to fulfill $\widetilde{P}(0) = W_2 P(0)$. Since

$$P(0)^{-1} = \frac{1}{6-\frac{1}{9}} \begin{bmatrix} 3 & -\frac{1}{3} \\ -\frac{1}{3} & 2 \end{bmatrix}$$

we choose

$$W_2 = \begin{bmatrix} 9 & -1 \\ -1 & 6 \end{bmatrix}$$

to get nice whole numbers to use. This gives

$$C = W_1 C_d W_2 = \begin{bmatrix} 9K_1 \dfrac{sT_1+1}{sT_1} & -K_2 \dfrac{sT_2+1}{sT_2} \\ -K_1 \dfrac{sT_1+1}{sT_1} & 6K_2 \dfrac{sT_2+1}{sT_2} \end{bmatrix}$$

**4 a.** The gain of the system equals the maximum of the largest singular value of $P(i\omega)$. From the plot in Figure 3 we see that the gain is approximately 1.14.

**b.** We have that

$$\begin{aligned} P(1i) &= \frac{1}{i^2+4i+5} \begin{bmatrix} i^2+3i+4 & -(i+3) \\ (i+3) & i^2+3i+4 \end{bmatrix} = \frac{1}{4+4i} \begin{bmatrix} 3+3i & -(3+i) \\ 3+i & 3+3i \end{bmatrix} \\ &= \frac{1-i}{8} \begin{bmatrix} 3+3i & -(3+i) \\ 3+i & 3+3i \end{bmatrix} = \begin{bmatrix} 0.75 & -0.5+0.25i \\ 0.5-0.25i & 0.75 \end{bmatrix} \end{aligned}$$

The middle singular value decomposition that was given matches this. The smallest amplification in $L_2$ norm at this frequency is given by the smallest singular value, which we see is 0.71.

The right singular vector corresponding to the smallest singular value is given by the second column of $V$ (not $V^*$ !), this is the conjugate of the second row of the last matrix in the singular-value decomposition, i.e., $[-0.71 \quad -0.71i]$. (Recall that the singular value decomposition is given by $A = U\Sigma V^*$.) This singular vector corresponds a vector-valued, sinusoidal signal where the phase of the second component is 90° ahead of the first component, i.e., we could for example take

$$u(t) = \begin{bmatrix} 0.71 \sin t \\ 0.71 \sin(t+\pi/2) \end{bmatrix}$$

(or any scaled/phase-shifted version of this).

**5 a.** From the block diagram, we have

$$\begin{aligned} HF(u-v) + KPv = u \qquad &\Rightarrow \qquad u = (I-HF)^{-1}(KP-HF)v \\ &\Rightarrow \qquad G = (I-HF)^{-1}(KP-HF). \end{aligned}$$

**b.** To be able to use the small gain theorem to conclude that a certain $\Delta_i$ gives a stable closed-loop system, we must have that $\Delta_i$ is stable and that

$$||\Delta_i|| < \frac{1}{||G||_\infty} = \frac{1}{2}.$$

It is clear that we obtain the largest value $|\Delta_1(i\omega)| \to 1/5$ as $\omega \to 0$, and since $\Delta_1$, also is stable, we can conclude closed-loop stability in this case.

The magnitude of $|\Delta_2(i\omega)| \to 1$ as $\omega \to \infty$, so the Small Gain Theorem cannot be used to conclude stability in this case.

Neither $\Delta_3$, nor $\Delta_4$ are stable, so the Small Gain Theorem is not applicable in these cases.

In summary, the Small Gain Theorem can be used to guarantee stability only for the feedback interconnection with $\Delta_1$.

**c.** The small gain theorem only gives a *sufficient* condition for closed-loop stability. Thus it is not possible to conclude that any of the uncertainty blocks gives an unstable closed-loop system. Even open-loop unstable systems can lead to stable closed-loop systems.

**6 a.** Let $S = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}$ be the solution to the Riccati equation

$$A^T S + SA - (SB)Q_2^{-1}B^T S + Q_1 = 0$$

where

$$Q_1 = CC^T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad Q_2 = 1.$$

The minimal cost of $J_a$ is then given by

$$x_0^T S x_0 = [0 \quad 1] S \begin{bmatrix} 0 \\ 1 \end{bmatrix} = s_3.$$

Writing out the Riccati equation we get

$$\begin{bmatrix} -0.5 & 1 \\ -1 & 0 \end{bmatrix} S + S \begin{bmatrix} -0.5 & -1 \\ 1 & 0 \end{bmatrix} - \left( S \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \cdot 1 \cdot [1 \quad 0] S + \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} = 0,$$

and with $S = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}$,

$$\begin{bmatrix} -0.5s_1 + s_2 & -0.5s_2 + s_3 \\ -s_1 & -s_2 \end{bmatrix} + \begin{bmatrix} -0.5s_1 + s_2 & -s_1 \\ -0.5s_2 + s_3 & -s_2 \end{bmatrix} - \begin{bmatrix} s_1^2 & s_1 s_2 \\ s_1 s_2 & s_2^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} = 0$$

$$\Leftrightarrow \begin{bmatrix} -s_1 + 2s_2 - s_1^2 & -0.5s_2 + s_3 - s_1 - s_1 s_2 \\ -0.5s_2 + s_3 - s_1 - s_1 s_2 & -s_2^2 - 2s_2 + 3 \end{bmatrix} = 0$$

from this we see that $s_2^2 - 2s_2 + 3 = 0$, so $s_2 = 1$ (we are looking for the positive definite solution of the Riccati equation). From $-s_1 + 2s_2 - s_1^2$, we get (using $s_2 = 1$) that $s_1 = 1$. Finally, from $-0.5s_2 + s_3 - s_1 - s_1 s_2 = 0$, we get that $s_3 = 0.5s_2 + s_1 + s_1 s_2 = 5/2$.

Thus, the minimal value of $J_a$ is $5/2$.

**b.** The controllability Gramian $W_c$ can be found by solving the Lyapunov equation

$$AW_c + W_cA^T + BB^T = 0$$

which reads

$$\begin{bmatrix} -0.5 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} w_1 & w_2 \\ w_2 & w_3 \end{bmatrix} + \begin{bmatrix} w_1 & w_2 \\ w_2 & w_3 \end{bmatrix} \begin{bmatrix} -0.5 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} -w_1 - 2w_2 + 1 & w_1 - 0.5w_2 - w_3 \\ w_1 - 0.5w_2 - w_3 & 2w_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solving for $W_c$, gives

$$W_c = \begin{bmatrix} w_1 & w_2 \\ w_2 & w_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

From Lecture 6 we get that

$$J_b = \int_0^\infty u(t)^2\, dt \geq x_1^T W_c^{-1} x_1 = \begin{bmatrix} 0 & 1 \end{bmatrix} W_c^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1.$$

**c.** The spectrum can be factorized as

$$\Phi_d(\omega) = H(i\omega)H(-i\omega)$$

with

$$H(s) = \frac{2}{s+0.1}.$$

Thus

$$D = \frac{2}{s+0.1} V$$

gives (for instance) the state-space realization

$$\dot{d} = -0.1d + 2v.$$

Augmenting the state-space description with this noise model gives

$$A = \begin{bmatrix} -0.5 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & -0.1 \end{bmatrix}, \quad B_u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad B_v = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & \sqrt{3} & 0 \end{bmatrix}, \quad D = 0$$

**7 a.** The closed-loop transfer function from $w = \begin{pmatrix} r \\ d \end{pmatrix}$ to $z = \begin{pmatrix} y \\ u \end{pmatrix}$ is

$$G_{zw} = \begin{pmatrix} \dfrac{PCF}{1+PC} & \dfrac{P}{1+PC} \\ \dfrac{CF}{1+PC} & -\dfrac{PC}{1+PC} \end{pmatrix}$$

**b.** Setting $Q = \dfrac{C}{1+PC}$ and using the fact that $\dfrac{P}{1+PC} = PS = P(1-T)$, this can be written

$$G_{zw} = \begin{pmatrix} PFQ & P(1-PQ) \\ FQ & -PQ \end{pmatrix}$$

This is an affine expression in $Q$.

**c.** The closed-loop transfer function from $r$ to $u$, setting $F = 1$, is

$$G_{ur} = \frac{C}{1+PC} = Q$$

Setting $Q = q_0 + \dfrac{q_1}{s+1}$, the requirement on $|G_{ur}(i)|$ is given by

$$\begin{aligned}
\left| q_0 + \frac{q_1}{i+1} \right| &\leq 2 \\
\frac{|q_0 i + q_0 + q_1|}{|i+1|} &\leq 2 \\
|q_0 i + q_0 + q_1|^2 &\leq 8 \\
q_0^2 + (q_0 + q_1)^2 &\leq 8
\end{aligned}$$

This is a quadratic constraint on $q_0$ and $q_1$.