LUND UNIVERSITY

Department of
AUTOMATIC CONTROL

FRTN10 Multivariable Control

Exam 2017-10-27, 14:00–19:00

Points and grades

All answers must include a clear motivation and a well-formulated answer. Answers may be given in English or Swedish. The total number of points is 25. The maximum number of points is specified for each subproblem.

Accepted aid

The textbook *Glad & Ljung*, standard mathematical tables like TEFYMA, an authorized "Formelsamling i Reglerteknik"/"Collection of Formulas" and a pocket calculator. Hand-outs of lecture notes and lecture slides (including markings/notes) are also allowed.

Results

The result of the exam will be entered into LADOK. The solutions will be available on the course home page: `http://www.control.lth.se/course/FRTN10`

1. Consider the following system:

$$G(s) = \begin{bmatrix} \dfrac{1}{(s+2)(s+3)} & \dfrac{1}{(s+3)(s+4)} & \dfrac{1}{(s+2)(s+4)} \end{bmatrix}$$

a. What are the poles and zeros of the system, and what are their multiplicity? (1.5 p)

b. Give a minimal state-space realization of the system. (2 p)

2. Four linear controllers have been designed for four different linear plants:

$$P_1(s) = \frac{12}{s(s+5)} \qquad P_2(s) = \frac{e^{-s}}{(s+1)^2}$$

$$P_3(s) = \frac{1-5s}{s(s+1)^2} \qquad P_4(s) = \frac{100}{(s+10)(s-10)}$$

The controllers are all minimum-phase, and no poles or zeros have been cancelled. Figure 1 shows four closed-loop step responses, and Figure 2 shows four Bode magnitude diagrams of the sensitivity and complementary sensitivity functions. Pair each plant P_1–P_4 with the corresponding closed-loop step-response plot A–D and the corresponding sensitivity plot I–IV. Motivate! (3 p)

3. Consider the following MIMO system:

$$P(s) = \begin{bmatrix} \dfrac{2}{s+1} & \dfrac{1}{s+3} \\ \dfrac{2}{s+6} & \dfrac{3}{s+1} \end{bmatrix}$$

We wish to control the process using two decentralized PI controllers,

$$C_d(s) = \begin{bmatrix} K_1 \dfrac{sT_1+1}{sT_1} & 0 \\ 0 & K_2 \dfrac{sT_2+1}{sT_2} \end{bmatrix}$$

Assuming the controller structure $C = W_1 C_d W_2$, decouple the system in stationarity and calculate the resulting controller C. (2 p)

4. Consider the linear system

$$P(s) = \frac{1}{s^2+4s+5} \begin{bmatrix} s^2+3s+4 & -(s+3) \\ s+3 & s^2+3s+4 \end{bmatrix}$$

A plot of the system's largest singular value is shown in Figure 3, and a number of singular value decompositions are given below.

a. What is the gain of the system? (1 p)

b. What is the *minimum* amplification of a signal with frequency $\omega = 1$ rad/s? Give an example of such an input signal in the time domain. (2 p)

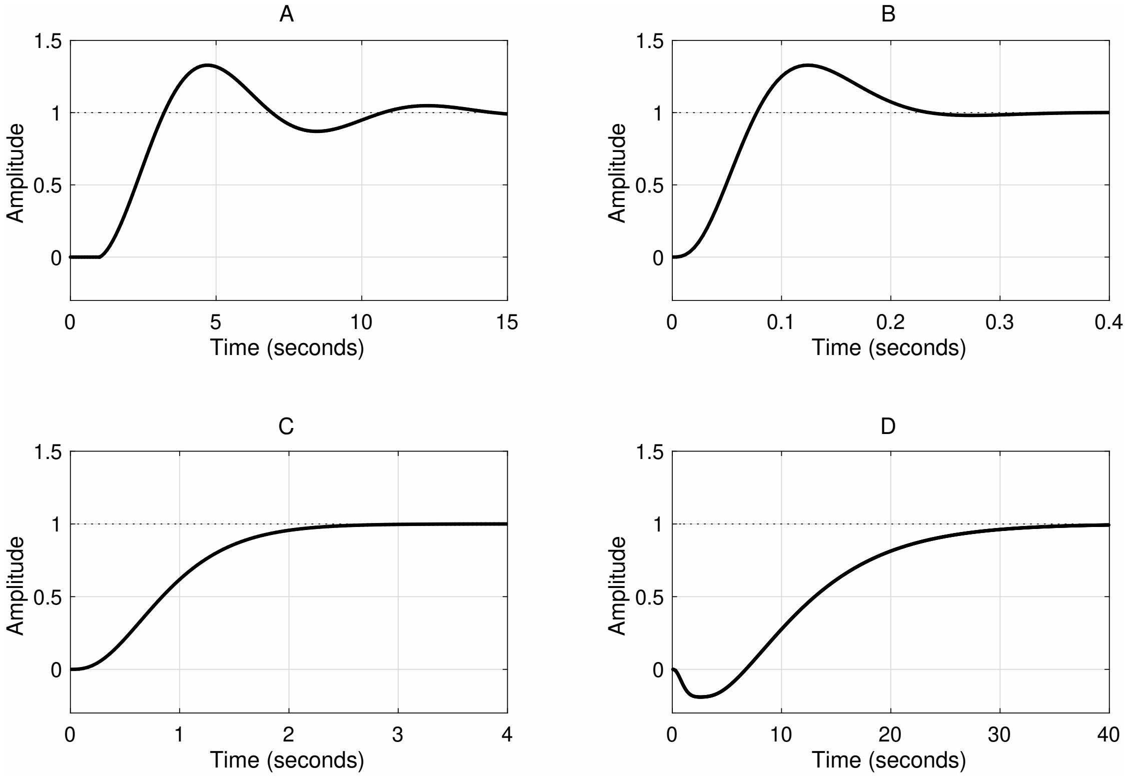

Figure 1 Closed-loop step responses in Problem 2.

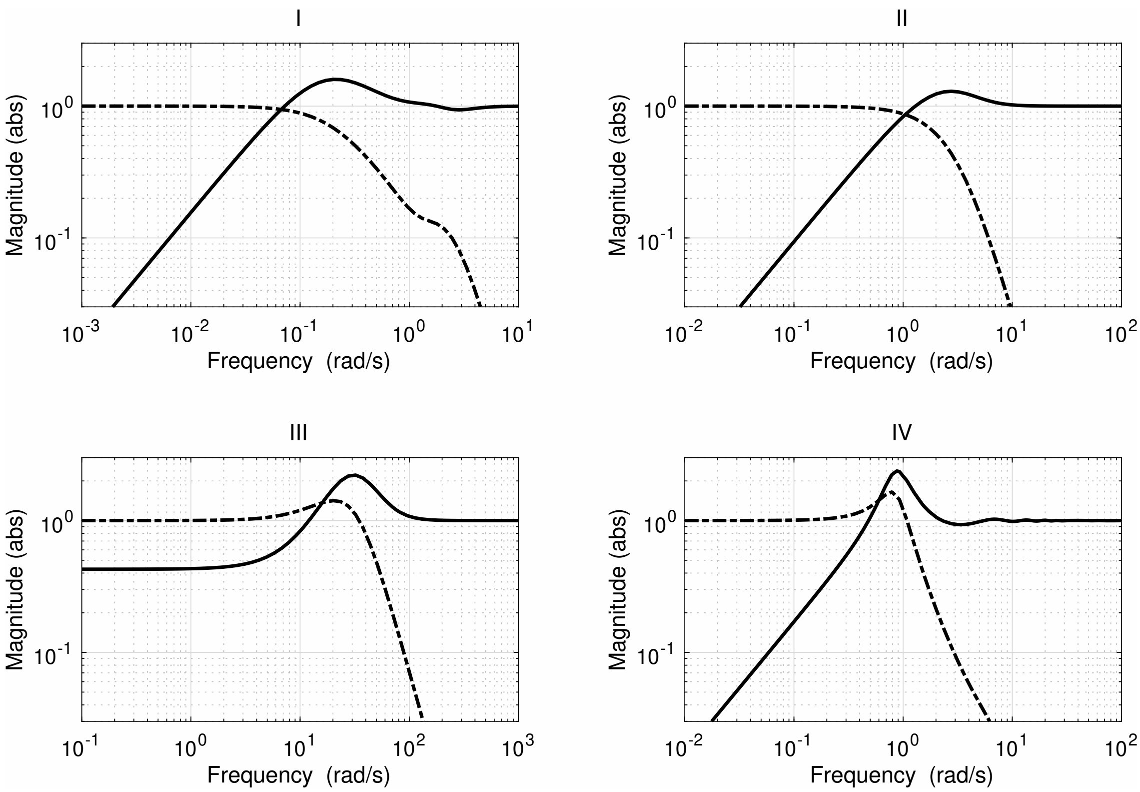

Figure 2 Sensitivity magnitude functions (full) and complementary sensitivity magnitude functions (dot-dashed) in Problem 2.

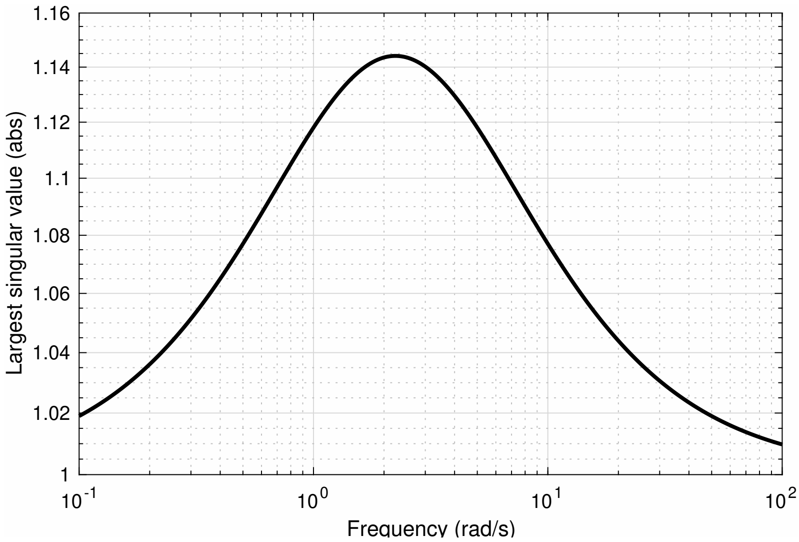

Figure 3 Plot of the largest singular value of $P(i\omega)$ in Problem 4.

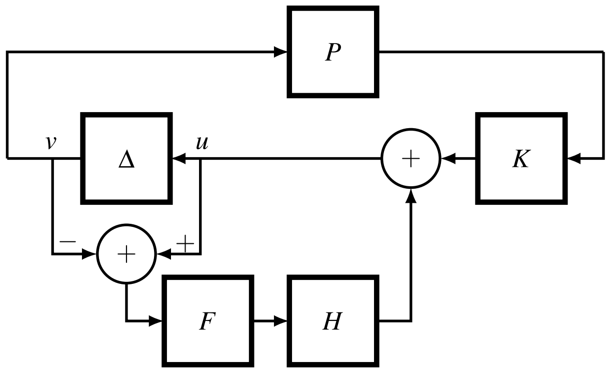

Figure 4 Control system in Problem 5.

The following singular value decompositions could be used in the subproblems above:

$$\begin{bmatrix} 0.74+0.09i & -0.29+0.34i \\ 0.29-0.34i & 0.74+0.09i \end{bmatrix} = \begin{bmatrix} -0.67-0.24i & -0.63+0.32i \\ -0.24+0.67i & -0.32-0.63i \end{bmatrix} \begin{bmatrix} 1.14 & 0 \\ 0 & 0.45 \end{bmatrix} \begin{bmatrix} -0.71 & -0.71i \\ -0.71 & 0.71i \end{bmatrix}$$

$$\begin{bmatrix} 0.75+0.00i & -0.50+0.25i \\ 0.50-0.25i & 0.75+0.00i \end{bmatrix} = \begin{bmatrix} -0.63-0.32i & -0.50+0.50i \\ -0.32+0.63i & -0.50-0.50i \end{bmatrix} \begin{bmatrix} 1.12 & 0 \\ 0 & 0.71 \end{bmatrix} \begin{bmatrix} -0.71 & -0.71i \\ -0.71 & 0.71i \end{bmatrix}$$

$$\begin{bmatrix} 0.90+0.30i & 0.70-0.10i \\ -0.70+0.10i & 0.90+0.30i \end{bmatrix} = \begin{bmatrix} -0.50-0.50i & -0.63+0.32i \\ 0.50-0.50i & 0.32+0.63i \end{bmatrix} \begin{bmatrix} 1.41 & 0 \\ 0 & 0.89 \end{bmatrix} \begin{bmatrix} -0.71 & 0.71i \\ -0.71 & -0.71i \end{bmatrix}$$

5. A control system is shown in the block diagram in Figure 4. We want to isolate the uncertainty block Δ as shown in Figure 5.

a. Find the transfer function G from v to u expressed in terms of P, K, H and F. Assume that all blocks are MIMO. (1 p)

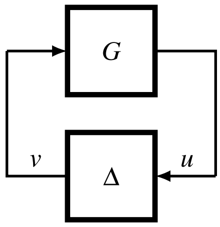

Figure 5 Simplified model of Figure 4.

b. Now assume that all blocks are SISO and that G is stable with gain $||G||_\infty = 2$. For which of the following Δ_i can you guarantee stability of the closed-loop system using the Small Gain Theorem? (2 p)

$$\Delta_1(s) = \frac{1}{s+5} \qquad \Delta_2(s) = \frac{s+1}{s+3}$$
$$\Delta_3(s) = \frac{0.1}{s} \qquad \Delta_4(s) = \frac{0.4}{s-1}$$

c. For which of the above uncertainty blocks Δ_i can you guarantee that the closed loop system is unstable? (0.5 p)

6. We will consider control of the damped oscillator in Figure 6, where $\zeta = 0.25$ is the damping ratio, and $\omega_0 = 1$ is the resonance frequency.

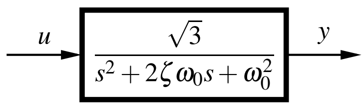

Figure 6 Model of a damped oscillator.

Use the following state-space representation of the system,

$$A = \begin{bmatrix} -0.5 & -1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & \sqrt{3} \end{bmatrix}, \qquad D = 0$$

a. Using optimal state feedback, $u = -Lx$, what is the minimum value of

$$J_a = \int_0^\infty \left(y^2 + u^2\right) dt$$

when starting from the state $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$? (2.5 p)

b. State a lower bound on the cost

$$J_b = \int_0^\infty u^2 \, dt$$

for driving the state from $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. (2 p)

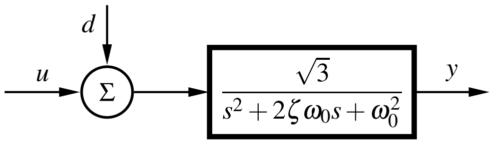

Figure 7 Model of a damped oscillator with input disturbance.

c. Assume that there is a low-frequency input disturbance d to the plant (Figure 7), and that its spectrum is given by

$$\Phi_d(\omega) = \frac{4}{\omega^2 + 0.01}.$$

Augment the state-space model to include a model for the disturbance d. (The input to the disturbance model is, as ususal, white noise with intensity 1.) (2 p)

7. Consider the control loop in Figure 8, where the plant $P(s)$ is a stable SISO system, $F(s)$ is a stable filter, and $C(s)$ is a controller to be optimized.

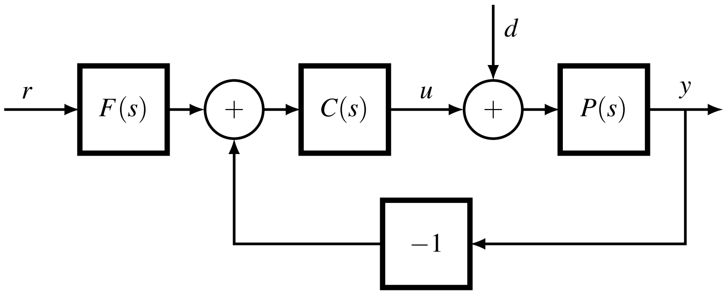

Figure 8 The control loop in Problem 7.

a. Find the closed-loop transfer matrix from $w = \begin{pmatrix} r \\ d \end{pmatrix}$ to $z = \begin{pmatrix} y \\ u \end{pmatrix}$. (1 p)

b. Introduce the Youla parameter

$$Q = \frac{C}{1+PC}$$

and show that the closed-loop transfer matrix from w to z can be written as an affine function of Q (which is suitable for optimization).

(*Hint:* You may want to use the fact that $S+T=1$.) (1 p)

c. For convex optimization, introduce the (very simple) representation

$$Q(s) = q_0 + \frac{q_1}{s+1}$$

where the scalar parameters q_0 and q_1 are the variables to be optimized.

Let $F(s) = 1$. Show that the closed-loop requirement

$$|G_{ur}(i\omega)| \le 2, \quad \omega = 1$$

can be expressed as a quadratic convex constraint. (1.5 p)