LUND UNIVERSITY

Department of
AUTOMATIC CONTROL

FRTN10 Multivariable Control

Exam 2017-01-03, 08:00–13:00

Points and grades

All answers must include a clear motivation and a well-formulated answer. Answers may be given in English or Swedish. The total number of points is 25. The maximum number of points is specified for each subproblem.

Accepted aid

The textbook *Glad & Ljung*, standard mathematical tables like TEFYMA, an authorized “Formelsamling i Reglerteknik”/”Collection of Formulas” and a pocket calculator. Handouts of lecture notes and lecture slides (including markings/notes) are also allowed.

Results

The result of the exam will be entered into LADOK. The solutions will be available on the course home page: `http://www.control.lth.se/course/FRTN10`

Solution to Exam in FRTN10 Multivariable Control 2016-10-25

1. Consider the following system:

$$G(s) = \begin{bmatrix} \dfrac{2}{(s+10)(s+1)} & \dfrac{1}{s+1} \\ \dfrac{2}{s+2} & \dfrac{1}{s+2} \end{bmatrix}$$

a. Determine the poles and zeros of the system, including their multiplicity. (2 p)

b. Write the system in state-space form using a minimal number of state variables. (1.5 p)

c. Compute the RGA of the system in stationarity. Which inputs should be paired with which outputs in a decentralized control design? (1.5 p)

Solution

a. The poles are determined by the smallest common denominator of the sub-determinants of $G(s)$. The sub-determinants are:

$$\frac{2}{(s+10)(s+1)}, \quad \frac{1}{s+1}, \quad \frac{2}{s+2}, \quad \frac{1}{s+2}, \quad \frac{-2(s+9)}{(s+10)(s+1)(s+2)}$$

Where the first four are the 1×1 sub-determinants of $G(s)$ and the last one is the full 2×2 determinant. The smallest common denominator among the sub-determinants is $(s+1)(s+2)(s+10)$, and the poles are thus located in $-1, -2$ and -10.

The zeros are determined by the largest common divisor of the nominators of the largest sub-determinants, normalized with the pole polynomial in the denominator. The largest sub-determinant in this case is the full determinant of $G(s)$, and since it is already has the pole polynomial as its denominator we immediately see that the process has a zero in -9.

b. We start by dividing $G(s)$ into separate terms using partial fraction decomposition, and then subdividing the matrix:

$$G(s) = \begin{bmatrix} \frac{-\frac{2}{9}}{s+10} + \frac{\frac{2}{9}}{s+1} & \frac{1}{s+1} \\ \frac{2}{s+2} & \frac{1}{s+2} \end{bmatrix} = \frac{1}{s+1}\begin{bmatrix} \frac{2}{9} & 1 \\ 0 & 0 \end{bmatrix} + \frac{1}{s+2}\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} + \frac{1}{s+10}\begin{bmatrix} -\frac{2}{9} & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{s+1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} \frac{2}{9} & 1 \end{bmatrix} + \frac{1}{s+2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \end{bmatrix} + \frac{1}{s+10}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} -\frac{2}{9} & 0 \end{bmatrix}$$

From this form it is straight-forward to obtain a diagonal state-space representation of the system (see notes from lecture 6):

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -10 \end{bmatrix} x(t) + \begin{bmatrix} \frac{2}{9} & 1 \\ 2 & 1 \\ -\frac{2}{9} & 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x(t)$$

c. We have:

$$G(0) = \begin{bmatrix} \frac{1}{5} & 1 \\ 1 & \frac{1}{2} \end{bmatrix} \quad G(0)^{-1} = -\frac{10}{9} \begin{bmatrix} \frac{1}{2} & -1 \\ -1 & \frac{1}{5} \end{bmatrix}$$

which gives the RGA:

$$\text{RGA}(G(0)) = G(0).*G(0)^{-T} = \begin{bmatrix} -\frac{1}{9} & \frac{10}{9} \\ \frac{10}{9} & -\frac{10}{9} \end{bmatrix}$$

Since we should avoid pairing of inputs and outputs which will result in negative diagonal elements in RGA$(G(0))$, the RGA matrix suggests that we should pair u_1–y_2 and u_2–y_1.

2. Design an Internal Model Controller for the process

$$P(s) = \frac{s+1}{(0.1s+1)^3}$$

Place the poles of the closed-loop system in the same location as the open-loop poles. Will the closed-loop system be able to follow a constant reference signal without stationary error? (3 p)

Solution

We can select e.g.

$$Q(s) = \frac{P^{-1}(s)}{(0.1s+1)^2} = \frac{0.1s+1}{s+1}$$

This gives the closed-loop system

$$Q(s)P(s) = \frac{1}{(0.1s+1)^2}$$

which has the same poles as the open-loop system. The controller is then given by

$$C(s) = \frac{Q(s)}{1 - Q(s)P(s)} = \frac{\frac{0.1s+1}{s+1}}{1 - \frac{1}{(0.1s+1)^2}} = \frac{0.1(s+10)^3}{s(s+1)(s+20)}$$

The controller has a pole in 0, i.e., an integrator, so the closed-loop system will be able to follow a constant reference signal without error.

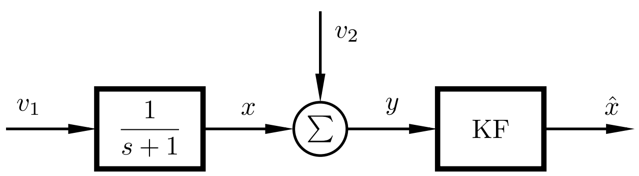

Figure 1 An open-loop system.

3. Consider the open-loop system in Figure 1. You should design the Kalman filter KF such that $\hat{x}$ is an optimal estimate of x. v_1 and v_2 are zero-mean white noise processes with intensities $R_1 = 6$ and $R_2 = 1$ respectively, and their cross-intensity is $R_{12} = 1$.

a. Show that the transfer function of the resulting Kalman filter is $\frac{2}{s+3}$. (2 p)

b. Calculate the stationary variance of x and the spectral density of x. (2 p)

Solution

a. A state-space realization of the process is given by

$$\dot{x} = -x + v_1$$
$$y = x + v_2$$

from which we identify $A = -1$, $N = C = 1$. The Kalman filter is given by

$$\dot{\hat{x}} = A\hat{x} + K(y - C\hat{x})$$

where $K = (PC + NR_{12})/R_2$, where $P > 0$ is given by the solution to the Riccati equation

$$2AP + R_1 - (PC + R_{12})^2/R_2 = 0$$

We obtain

$$(P+1)^2 + 2P - 6 = 0 \quad \Rightarrow \quad P = 1 \quad \Rightarrow \quad K = 2$$

Taking the Laplace transform of the Kalman filter equation and solving for $\hat{X}$ we obtain

$$\hat{X}(s) = \frac{2}{s+3}Y(s)$$

b. Let $\pi = \mathrm{E}\, x^2$. We have the Lyapunov equation

$$-1 \cdot \pi - \pi \cdot 1 + 6 = 0$$

with the solution $\pi = 3$. The variance of x is hence 3.

The spectral density of x is given by

$$\phi_x(\omega) = R_1 \frac{1}{1 + i\omega} \frac{1}{1 - i\omega} = \frac{6}{1 + \omega^2}$$

4. A cascade control system is shown in the block diagram in Figure 2. We want to isolate the uncertainty as shown in Figure 3.

a. Find the transfer function G from n to u_2 expressed in terms of C_1, C_2, P_1, P_2 and F. (1 p)

b. The step response and the singular value plot of G are shown in Figures 4 and 5. For which of the following $\Delta(s)$ can you guarantee stability of the closed-loop system using the Small Gain Theorem?

- $\Delta_1(s) = \dfrac{2}{s+5}$
- $\Delta_2(s) = 0.8$
- $\Delta_3(s) = \dfrac{0.3s + 1}{s+1}$

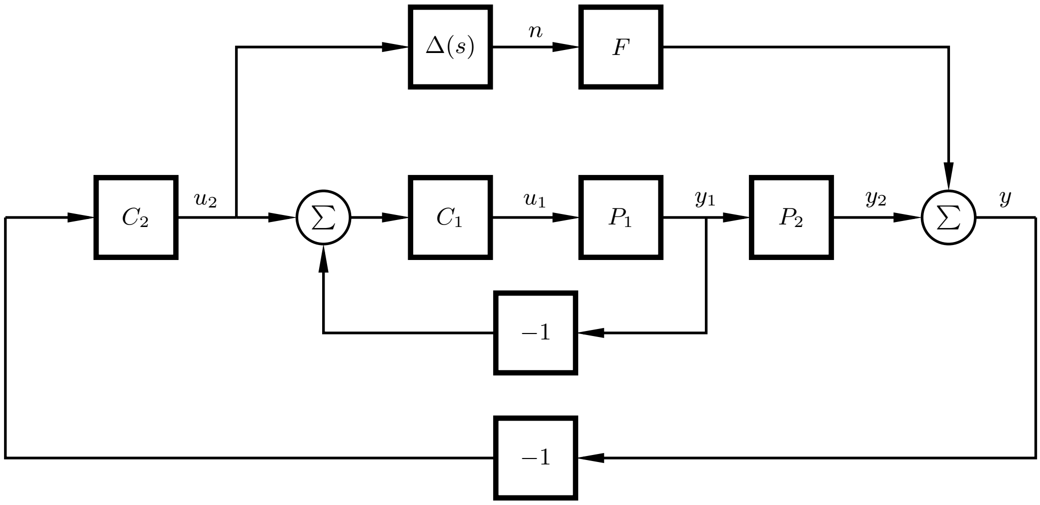

Figure 2 Block diagram for the system in Problem 4.

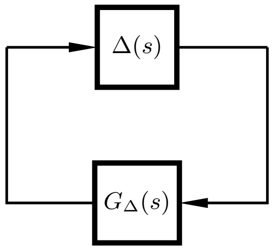

Figure 3 Desired block diagram for the system in Problem 4.

(1.5 p)

c. Is it possible that all of $\Delta_1(s)$, $\Delta_2(s)$ and $\Delta_3(s)$ can actually result in stable closed-loop systems? Motivate your answer. (0.5 p)

Solution

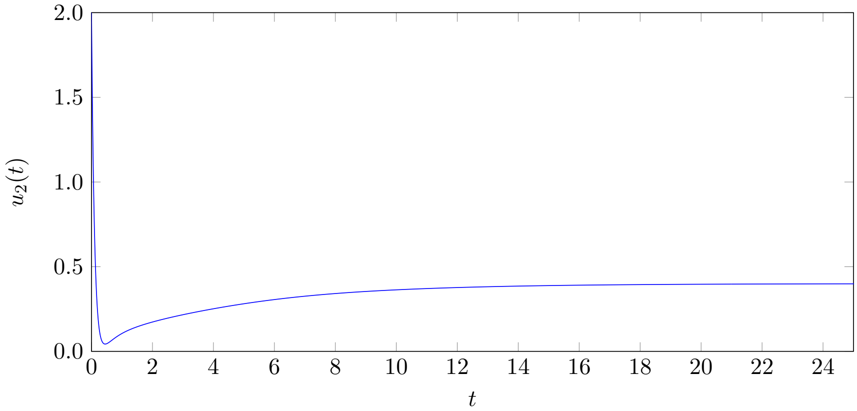

Figure 4 Step response of G.

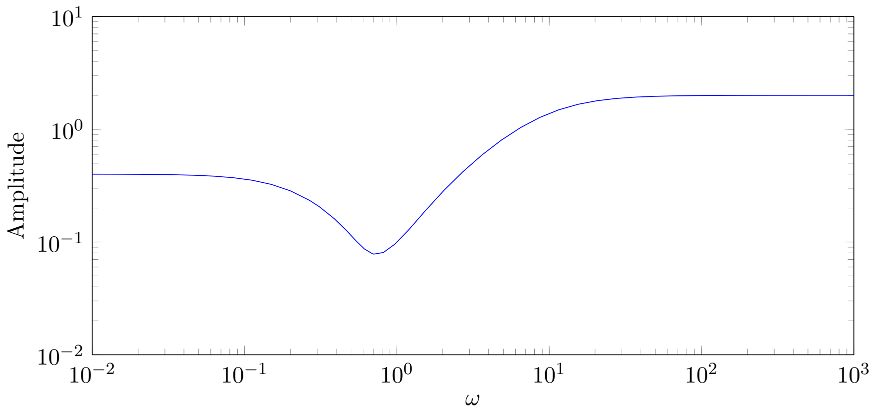

Figure 5 Singular value plot of G

a. To simplify we can look at some small parts first:

$$u_2 = -C_2 y$$
$$y = Fn + P_2 y_1$$
$$y_1 = \frac{P_1 C_1}{1 + P_1 C_1} u_2$$

Putting this together we get

$$u_2 = \frac{(1 + P_1 C_1) C_2 F}{1 + P_1 C_1 (1 + P_2 C_2)} n \Rightarrow G = \frac{(1 + P_1 C_1) C_2 F}{1 + P_1 C_1 (1 + P_2 C_2)}$$

b. As can be seen in the step response, G is stable, and from the sigma plot we see from the maximum singular value is $\|G\|_\infty = 2$. According to the Small Gain Theorem we can then guarantee stability for all $\Delta(s)$ such that $\|\Delta(s)\|_\infty < 1/2$. Since $\|\Delta_1(s)\| = 0.4$, stability can be guaranteed for that one. However, $\|\Delta_2(s)\|_\infty = 0.8$ and $\|\Delta_3(s)\|_\infty = 1$ so stability can not be guaranteed for those.

c. Yes. It is possible that also $\Delta_2(s)$ and $\Delta_3(s)$ could result in a stable closed loop, since the Small Gain Theorem is conservative.

5. You want to design an optimal controller that minimizes the cost function

$$J = \int \left(y^T(t) Q_1 y(t) + u^T(t) Q_2 u(t) \right) dt$$

with the weight matrices $Q_1 = \begin{pmatrix} 10 & 0 \\ 0 & 1 \end{pmatrix}$ and $Q_2 = 1$.

a. How many inputs u and outputs y does the system have? (0.5 p)

b. Explain in words how the closed-loop system behavior would change if the weight matrices were instead set to $Q_1^* = \begin{pmatrix} 1 & 0 \\ 0 & 0.1 \end{pmatrix}$ and $Q_2^* = 1$. (0.5 p)

c. The process is given by

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ \sqrt{10} \end{pmatrix} u$$
$$y = x$$

Design a state feedback law $u = -Lx$ that minimizes the cost function with the weight matrices Q_1 and Q_2. (1.5 p)

d. The result from subproblem c gives an optimal controller. But as you hopefully know, "optimal" is not the same as "good"; it depends on whether the cost function has been chosen wisely. With the given cost function, could you at least guarantee that the obtained closed-loop system will be stable? Motivate! (0.5 p)

Solution

a. The dimensions of Q_1 and Q_2 give that the system has one input and two outputs.

b. The relation between the first and second output are unchanged; it's just a scaling factor. The punishment on the control signal is however much higher with the starred weight matrices, so the control signal will be smaller.

c. The state feedback vector $L = Q_2^{-1}B^TS$ where S is the positive semidefinite solution to the Riccati equation $A^TS + SA + C^TQ_1C - SBQ_2^{-1}B^TS = 0$. Putting in the system matrices we end up with the equation system

$$\begin{aligned} 10 - 10s_2 &= 0 \\ s_1 - 10s_2s_3 &= 0 \\ 2s_2 + 1 + 10s_3 &= 0 \end{aligned}$$

which gives

$$S = \begin{pmatrix} 10\sqrt{3/10} & 1 \\ 1 & \sqrt{3/10} \end{pmatrix}$$

and $L = \begin{pmatrix} \sqrt{10} & \sqrt{3} \end{pmatrix}$

d. Yes we can! The LQ technique that we used always guarantees a stable system with a phase margin of at least 60 degrees. (If it would have been an LQG controller we could however not give this kind of guarantee.)

6. You have been given the task to design an optimal controller $C(s)$ for a stable, single-input–single-output system $P(s)$, assuming a standard 1-degree-of-freedom controller structure. The controller should minimize the integrated error of the output signal y_{refstep} due to a reference step,

$$\int_0^\infty |y_{\text{refstep}}|dt.$$

subject to the following constraints,

1. The system should be robust to process variations, i.e. $|S(i\omega)| \le 1.5$.

2. The impact of measurement noise on the control signal should be limited, i.e. $\sup_\omega |G_{un}(i\omega)| \leq 30$.
3. There should be less than 2% overshoot in the process output signal due to a reference step.
4. The control signal due to a reference step should be within $[-10, 10]$.

Since the plant is SISO and stable, the Youla parametrization

$$Q = C/(1 + PC), \tag{1}$$

has been chosen. For example, the following shows the affine relation between S and Q:

$$S = \frac{1}{1+PC} = 1 - \frac{PC}{1+PC} = 1 - PQ$$

A suitable finite basis $\{Q_k\}_{k=1}^{50}$ for the parametrization of Q was selected, and step responses and frequency responses for $\{Q_k\}$, $\{PQ_k\}$ were computed for the different basis functions at suitably chosen time points $\{t_1, t_2, \ldots, t_N\}$ and frequency points $\{\omega_1, \omega_2, \ldots, \omega_M\}$.

The results are have been stored in the Matlab variables `Q_sr`, `Q_fr`, `PQ_sr` and `PQ_fr`, where the columns correspond to different basis functions Q_k and the rows correspond to different time/frequency points. The variable b selects how much to use of each basis function, i.e. $Q = \sum Q_k b_k$.

Before falling asleep in front of the computer last night, you had started to code up the problem in `cvx` and had managed to specify the cost function and first constraint. The code that you woke up to was

```
M_s = 1.5;
max_overshoot = 1.02;
CS_max = 30;
umax = 10;

cvx_begin
variable b(50)

minimize sum( abs(ones(N,1) - PQ_sr*b ) )

subject to:
% 1. Maximum sensitivity constraint
max(abs(1 - PQ_fr*b)) <= M_s;

% 2. Constraint on transfer functionnnnnnnnnnnnnnnnnnnnnnnnnnnnnnn

cvx_end
```

a. What is the missing code, required to enforce the constraints 2–4? You do not have to get the `cvx`/Matlab syntax exactly right. (2 p)

b. The convex optimization approach discussed in this problem makes it very easy to design controllers which are very close to optimal, for a wide range of different objective functions and constraints. Give one reason why the controllers designed with this method can be problematic for real-world implementation. (0.5 p)

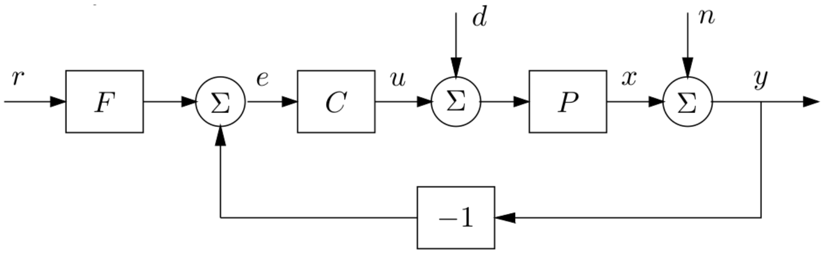

Figure 6 Two-degree-of-freedom controller structure for problem .

Solution

a. The missing lines should be along the lines of

```
% Constraint on transfer function n -> u
abs(Q_fr*b) <= CS_max

% Constraint on overshoot in y from reference step
PQ_sr*b <= max_overshoot;

% Constraint on control signal u for reference step
-umax <= Q_sr*b <= umax;
```

b. Either of the following answers are acceptable:

1. The controllers can have very high order, which makes it computationally expensive and numerically challenging to implement them.
2. The designed controllers can be unstable which is not desirable in real-world applications.

7. Let's do some loop shaping! The process is given by

$$P(s) = \frac{e^{-s}}{(s+1)(s+2)},$$

and the two-degree-of-freedom controller structure in Figure 6 is used to control it.

Your colleague has already designed a PID controller

$$C_1(s) = 1 + \frac{0.2}{s} + 0.2s.$$

The gang of four for this controller is plotted in Figure 7 and the effect of a step disturbance and measurement noise is plotted in Figure 8.

a. Mention one significant advantage of loop shaping compared to LQG when doing control design for single-input–single-output systems. (0.5 p)

b. As seen in Figure 8 there is significant control signal activity due to measurement noise. Why is this typically bad? How should you change the controller to reduce the impact of measurement noise on the control signal? (1 p)

c. As seen in Figure 8, a step disturbance in d is attenuated too slowly. How should you change the controller so that the disturbances are rejected faster? Mention two alternatives. (1 p)

d. Can you conclude only from Figure 7 that the closed-loop system is stable? Motivate your answer. (0.5 p)

e. Is it typically best to design the controller C or the prefilter F first? Motivate! (0.5 p)

f. One way to design the prefilter F is to choose

$$F = \frac{1 + PC}{PC(1 + sT_f)^d}$$

where d is chosen large enough to make F proper. Why will this approach not work in our case? How could the approach be modified to make it work? (1 p)

Solution

a. It is easier to take closed loop robustness into account when doing loop shaping, when doing LQG design there are no robustness guarantees.

b. High control signal activity tends to wear out the actuator or make actuator nonlinearities more noticeable. To reduce the impact of the measurement noise on the control signal, a low pass filter should be added to the controller.

c. To increase the speed for which load disturbances are rejected there are a few different options: increase the integral action, add a lag filter at low frequencies, or increase the controller gain/system bandwidth.

If we increase the integral action and add a low pass filter we get the following controller

$$C_2(s) = \left(1 + \frac{0.8}{s} + 0.2s\right) \Big/ (s/10 + 1)^2.$$

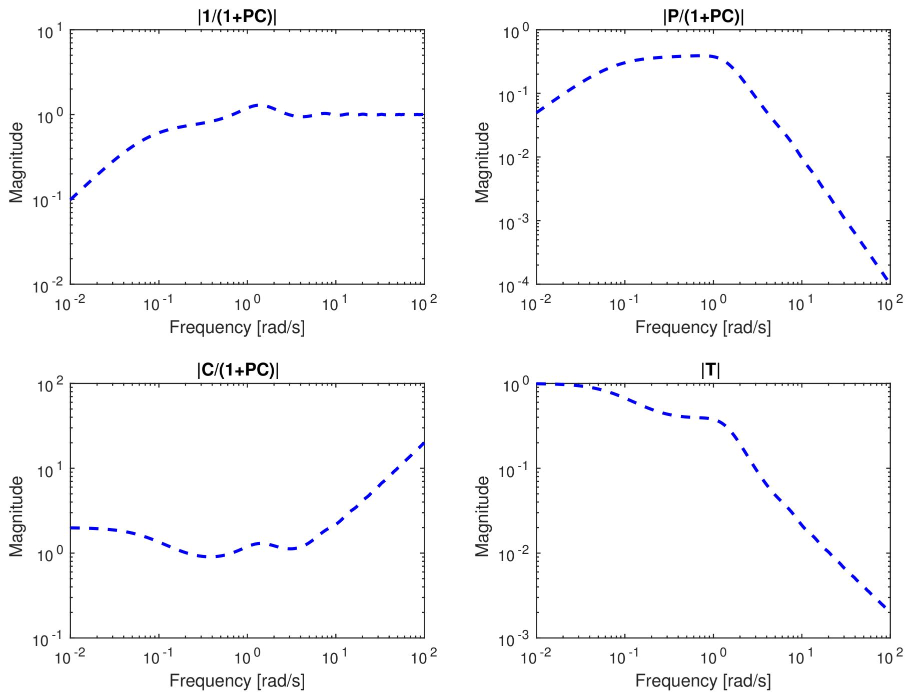

Figure 7 Gang of four for Problem .

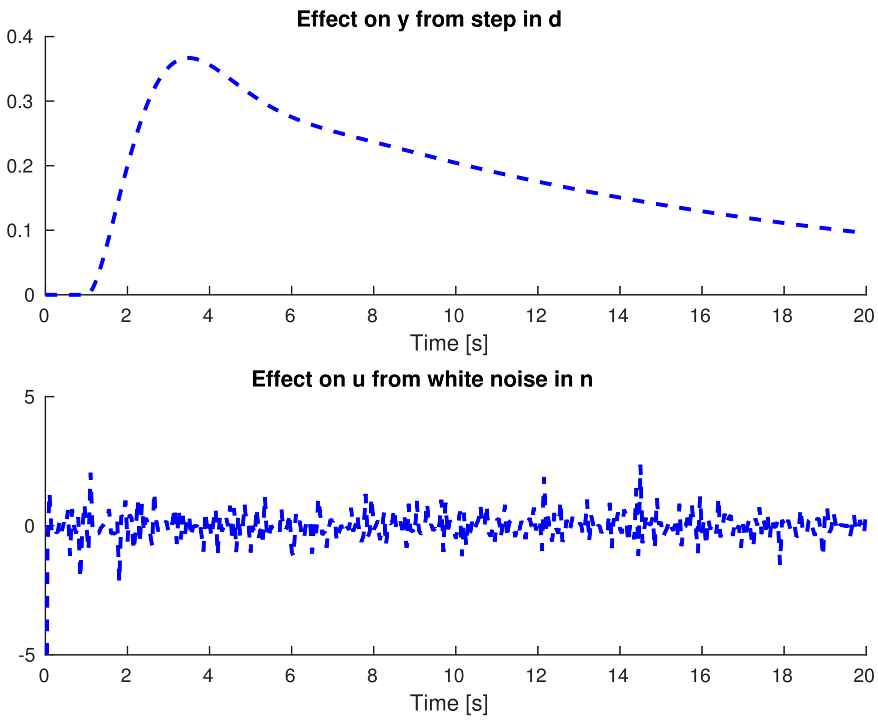

Figure 8 Effect of a step disturbance on the measured signal y and effect of white measurement noise on the control signal u, for Problem .

The gang of four and the time-domain responses for the improved controller are shown in Figures 9 and 10.

d. It is not possible to conclude stability of the closed loop system only from the magnitude plots in the gang of four. For example the magnitude plots of the unstable system $1/(s-1)$ and the stable system $1/(s+1)$ are the same.

e. The design of F does not impact robustness and disturbance rejection, so it is typically best to first design the controller C for good robustness and disturbance rejection, and then design the prefilter F for a good reference step response. If F would have been design first, the design of C would affect both robustness, disturbance rejection and the reference step response, which would have made things more complicated.

f. Since the plant has a time delay, $\frac{1+PC}{PC}$ will not be causal, and this cannot be helped by increasing d. To remedy the problem, the delay must be included in F, e.g.

$$F = \frac{(1+PC)e^{-s}}{PC(1+sT_f)^d}$$

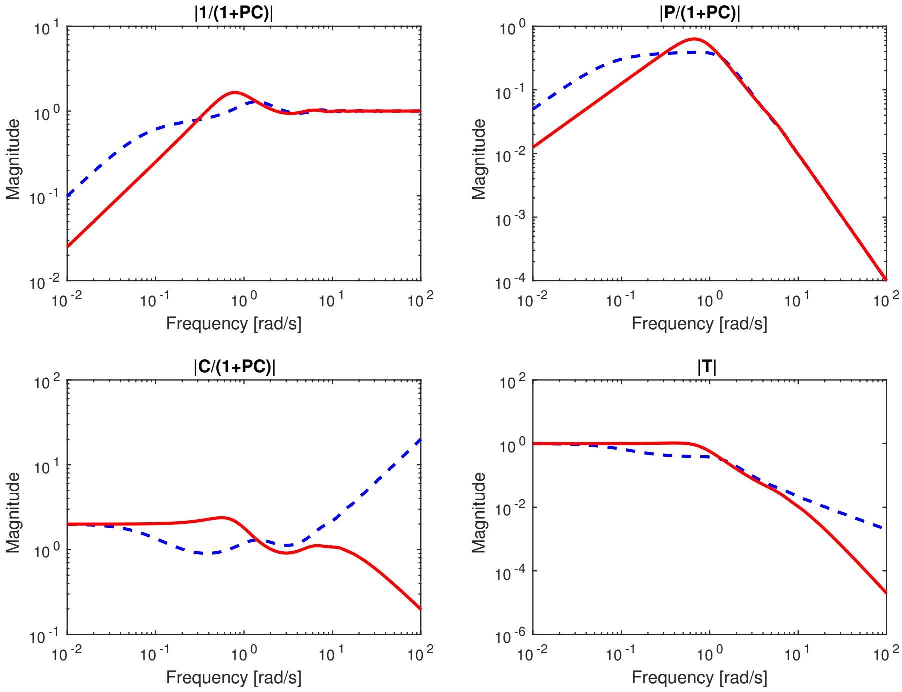

Figure 9 Gang of four for the system in Problem . The blue dashed line is for the original controller $C_1(s)$ and the red solid line is for the improved controller $C_2(s)$.

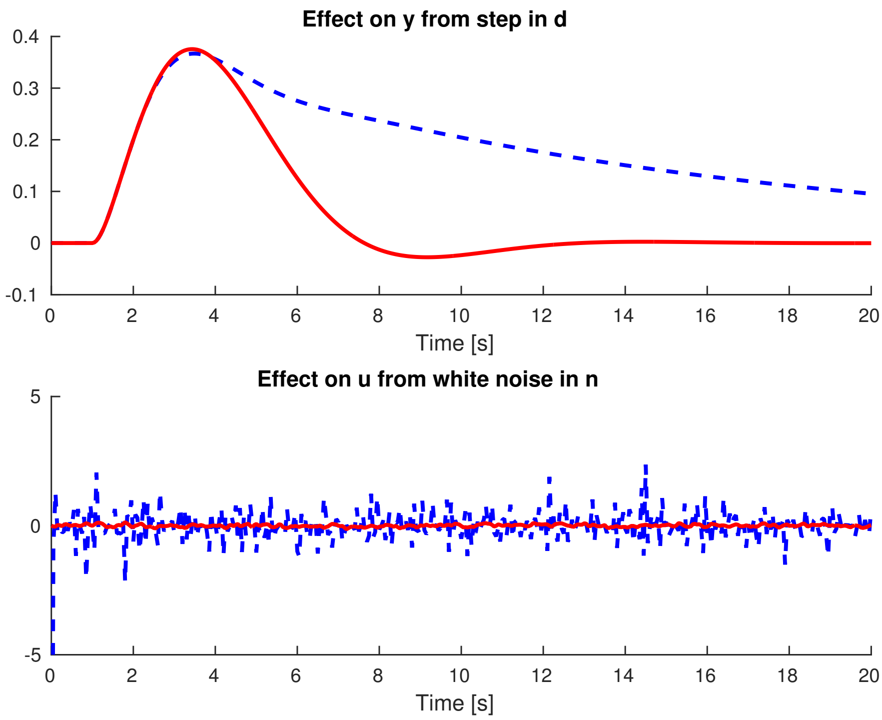

Figure 10 Effect of a step disturbance on the measured signal y and effect of white measurement noise on the control signal u, for Problem . The blue dashed line is for the original controller $C_1(s)$ and the red solid line is for the improved controller $C_2(s)$.