LUND UNIVERSITY

Department of
AUTOMATIC CONTROL

FRTN10 Multivariable Control

Exam 2016-10-25, 08:00–13:00

Points and grades

All answers must include a clear motivation and a well-formulated answer. Answers may be given in English or Swedish. The total number of points is 25. The maximum number of points is specified for each subproblem.

Accepted aid

The textbook *Glad & Ljung*, standard mathematical tables like TEFYMA, an authorized "Formelsamling i Reglerteknik"/"Collection of Formulas" and a pocket calculator. Hand-outs of lecture notes and lecture slides (including markings/notes) are also allowed.

Results

The result of the exam will be entered into LADOK. The result as well as solutions will be available on the course home page:
`http://www.control.lth.se/course/FRTN10`

Solution to Exam in FRTN10 Multivariable Control 2016-10-25

1. Let

$$G(s) = \begin{pmatrix} \dfrac{s}{s+1} & \dfrac{-2(s+1)}{s+2} \\ 3 & \dfrac{1}{s+2} \end{pmatrix}$$

a. Compute the poles of $G(s)$. Also state their multiplicity. (1 p)

b. Compute the (transmission) zeros of $G(s)$. Do they impose any fundamental performance limitations? (1 p)

Solution

a. The 1×1 minors of $G(s)$ are

$$\frac{s}{s+1}, \quad \frac{-2(s+1)}{s+2}, \quad 3, \quad \frac{1}{s+2}$$

and the 2×2 determinant is

$$\frac{s}{s+1} \cdot \frac{1}{s+2} - 3 \cdot \frac{-2(s+1)}{s+2} = \frac{6s^2 + 13s + 6}{(s+1)(s+2)}$$

The least common denominator all subdeterminants is $(s+1)(s+2)$, thus the poles of $G(s)$ are -1 and -2, both with multiplicity 1.

b. The maximal minor is the 2×2 determinant $\dfrac{6s^2 + 13s + 6}{(s+1)(s+2)}$. The zeros are given by the roots of the numerator polynomial, which are

$$s = -\frac{13}{12} \pm \frac{5}{12}$$

giving the zeros $-3/2$ and $-2/3$. The zeros are not in the RHP and hence do not impose any fundamental limitations.

2. Consider the stable system

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ -0.1 & -5 \end{pmatrix} x + \begin{pmatrix} 6 & 2 \\ 0 & 1 \end{pmatrix} u$$
$$y = Cx$$

a. Calculate the controllability Gramian S_x of the system. Is the system controllable? Which state is more difficult to control? (2 p)

b. The observability Gramian of the system is

$$O_x = \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix}$$

Calculate the Hankel singular values of the system. Which state would you truncate and how large could the error

$$\frac{||y - y_r||_2}{||u||_2}$$

become if we were to keep the static properties of the system, but reduce it to first order? Here, y_r denotes the output signal from the reduced order system. (2 p)

Solution

a. The controllability Gramian is the symmetric 2×2 matrix S_x that solves the Lyapunov equation $AS_x + S_x A^T + BB^T = 0$. Straightforward calculations give

$$S_x = \begin{pmatrix} 20 & 0 \\ 0 & 0.1 \end{pmatrix}$$

The system is controllables since the controllability Gramian is non-singular. The second state is much harder to control than the first one, the second eigenvalue of S_x being 200 times smaller than the first one.

b. We have

$$S_x O_x = \begin{pmatrix} 20 & 0 \\ 0 & 1 \end{pmatrix}$$

Since $S_x O_x$ is diagonal, the squared Hankel singular values are immediately given by the diagonal elements, implying $\sigma_1 = \sqrt{20}$ and $\sigma_2 = 1$. You would keep the state corresponding to the larger singular value, i.e. x_1. The error bound is

$$\frac{||y - y_r||_2}{||u||_2} \leq 2\sigma_2 = 2$$

if we reduce the system to first order.

3. A commonly used multivariable model of a distillation column was derived by Wood and Berry in 1973. The model is

$$G(s) = \begin{pmatrix} \dfrac{12.8e^{-s}}{1 + 16.7s} & \dfrac{-18.9e^{-3s}}{1 + 21s} \\ \dfrac{6.6e^{-7s}}{1 + 10.9s} & \dfrac{-19.4e^{-3s}}{1 + 14.4s} \end{pmatrix}$$

a. Calculate the Relative Gain Array in stationarity and decide how the input–output pairing should be done for decentralized control. Will there be visible interaction between the two control loops? (2 p)

b. Find suitable matrices W_1 and W_2 such that $\tilde{G} = W_2 G W_1$ is decoupled in stationarity. (1 p)

c. For each of the four statements below, explain whether it is true or false. (2 p)

i. After the decoupling in subproblem **b**, there will be no visible cross-coupling in the closed-loop step responses.

ii. By finding suitable decoupling matrices W_1 and W_2 we modify our physical process so that it becomes diagonal.

iii. The fundamental limitations imposed by the process deadtimes can be removed by properly chosen decoupling matrices.

iv. The decoupling approach allows us to design SISO controllers while still taking the MIMO process information into account.

Solution

a. The static gain matrix is

$$G(0) = \begin{pmatrix} 12.8 & -18.9 \\ 6.6 & -19.4 \end{pmatrix}$$

from which the RGA in stationarity becomes

$$G(0) \,.\!*\, G(0)^{-T} = \begin{pmatrix} 2.0094 & -1.0094 \\ -1.0094 & 2.0094 \end{pmatrix}$$

From this we can conclude that we should couple $u_1 \leftrightarrow y_1$ and $u_2 \leftrightarrow y_2$. However, the RGA is far from an identity matrix, so there will be visible interation.

b. The matrices could for instance be chosen as $W_1 = I$ and $W_2 = G(0)^{-1}$.

c.
i. False! $\tilde{G}$ is only decoupled in stationarity, so all dynamic behavior will still give rise to cross-couplings. Also, the outputs of the closed loop system G is not necessarily decoupled just because the outputs of $\tilde{G}$ are. This we saw in for instance lab 2, where the tank levels were still coupled, but the sum and difference of them were decoupled.

ii. False! With the decoupler we do not change the physical process at all, just the control structure.

iii. False! Fundamental limitations cannot be removed by any controller structure.

iv. True. The decentralized structure means that we design SISO controllers, and the decoupler takes the MIMO cross-coupling in stationarity into account.

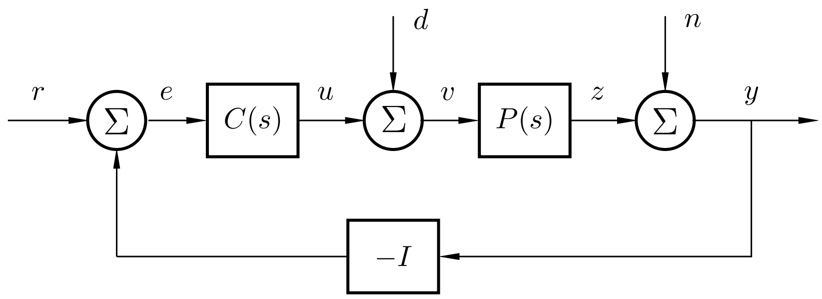

Figure 1 MIMO control system in Problem 4.

4. Consider the MIMO control system in Figure 1, where P and C are the (matrix) transfer functions of the process and the controller, respectively.

a. Calculate the (matrix) transfer function from $\begin{pmatrix} r \\ d \end{pmatrix}$ to u in terms of P and C. (1 p)

b. Suppose that $P(s)$ has 2 inputs and 3 outputs. What dimensions must then r, d and $C(s)$ have? (1 p)

c. The singular value (sigma) plots of P and C are shown in Figure 2. Can stability of the closed-loop system be guaranteed using the Small Gain Theorem? (1 p)

Solution

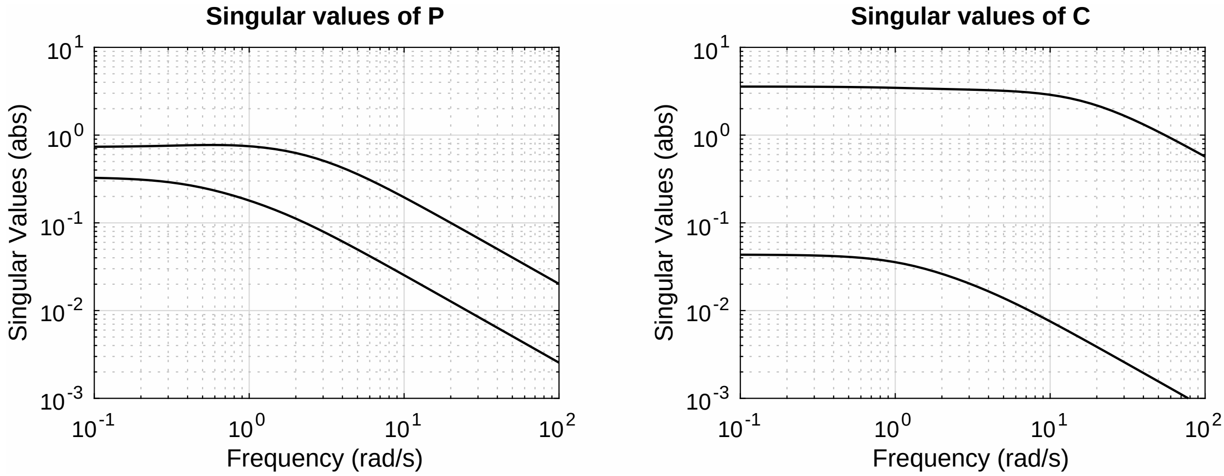

Figure 2 Sigma plots in Problem 4c.

a. From the block diagram, setting $n = 0$, we see that

$$U = C(R - P(D + U))$$

Solving for U we obtain

$$U = \begin{pmatrix} (I + CP)^{-1}C & -(I + CP)^{-1}CP \end{pmatrix} \begin{pmatrix} R \\ D \end{pmatrix}$$

b. $C(s)$ must have 3 inputs and 2 outputs, r must be a vector of size 3 and d must be a vector of size 2.

c. No. The gain of each system is given by the maximum of the largest singular value. The gain of P is larger than 0.7 and the gain of C is larger than 3. The loop gain is hence larger than 1 and stability can not be asserted using the Small Gain Theorem.

5. A young student who has only taken a basic course in control has attempted to design a controller for the process

$$P(s) = \frac{2 - s}{s(s^2 + 5s + 12)}$$

The controller $C(s)$ was designed as a state feedback from an observer. The control poles were placed in $-7 \pm i$ and -8 and the observer poles in $-14 \pm 2i$ and -16. The student proudly proclaims: “Look at these poles! I have designed a very fast and well-damped closed-loop system!”.

You become suspicious and ask the student to plot the magnitude of the sensitivity function S. The result in shown in Figure 3.

a. The student has forgotten all about the sensitivity function and its interpretation. Explain to him/her why this sensitivity function is a sign of very poor robustness. Also explain how you could immediately realize that it should not be possible to design a very fast closed-loop system for this plant. (2 p)

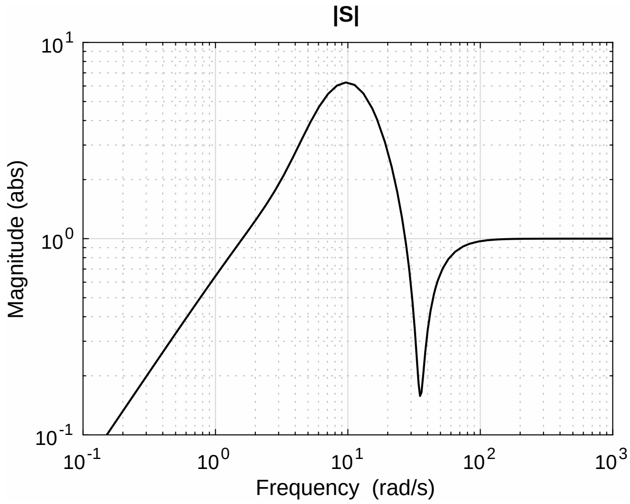

Figure 3 Magnitude plot of the sensitivity function in Problem 5.

b. Using the sensitivity weighting function

$$W_s(s) = \frac{s + M_s\omega_0}{M_s s}, \quad M_s,\, \omega_0 > 0$$

show that the specification

$$|S(i\omega)| \leq |W_s^{-1}(i\omega)|, \quad \forall\omega$$

is impossible to fulfill for any value of M_s if $\omega_0 > 2$. For what values of ω_0 is it impossible to fulfill $M_s \leq 1.4$? (2 p)

Solution

a. The inverse of the maximum sensitivity, $1/M_s$, measures the minimum distance between the Nyquist curve and the critical point -1. Here, $M_s \approx 6$, implying that the robustness is poor. The guaranteed amplitude margin is only $A_m = \frac{M_s}{M_s - 1} = 1.2$.

The plant has a non-minimum-phase (NMP) zero in 2. The rule of thumb for unstable zeros then says that the bandwidth of the closed-loop system cannot be faster than 2 rad/s (and should not be faster than 1 rad/s to ensure $M_s \leq 2$).

b. The Maximum Modulus Theorem implies that the specification $\|W_s S\|_\infty \leq 1$ is impossible to fulfill if $|W_s(z)| > 1$, where z is the location of the NMP zero. We have

$$|W_s(2)| = \frac{2 + M_s\omega_0}{2M_s}$$

which is always greater than 1 if $\omega_0 \geq 2$.

With $M_s = 1.4$ it is impossible to fulfill the specification if

$$|W_s(2)| = \frac{2 + 1.4\omega_0}{2 \cdot 1.4} > 1 \quad \Rightarrow \quad \omega_0 > 0.57$$

6. A controller derived from the standard LQG framework will not automatically feature integral action. One way of approximating it is to add a noise model to the Kalman filter where the noise is assumed to have a very high spectral density for low frequencies. Assuming that the initial model of the system is

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + v_1(t)\\ z(t) &= Cx(t)\\ y(t) &= Cx(t) + v_2(t)\end{aligned}$$

we would like to extend it to

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + v_1(t)\\ z(t) &= Cx(t) + w(t)\\ y(t) &= Cx(t) + v_2(t) + w(t)\end{aligned}$$

where w is noise with high spectral density for low frequencies. We model w as white noise n filtered through $H(s) = \frac{1}{s+\delta}$, i.e.

$$w = Hn$$

Here, $\delta > 0$ is some small number.

a. Explain why we do not use a pure integrator, i.e. $H(s) = \frac{1}{s}$, in our noise model when designing an LQG controller. (1 p)

b. Extend the process model with the noise model, such that it attains the form

$$\begin{aligned}\dot{x}_e(t) &= A_e x_e(t) + B_e u(t) + v_{1e}(t)\\ z(t) &= C_e x_e(t)\\ y(t) &= C_e x_e(t) + v_2(t)\end{aligned}$$

where $v_{1e} = [v_1^T \quad n]^T$. Express A_e, B_e and C_e in A, B and C. (1 p)

c. Now assume that $A = -1$, $B = 1$, $C = 1$ and $\delta = 0.001$. Which one of the three controllers A, B or C in Figure 4 could be the transfer function of an LQG controller based on the extended model? You can assume that the low- and high-frequency asymptotes of the controllers are visible in the plot. (1 p)

Solution

a. We would ideally like to have a noise model which has an infinite amplification for static signals, i.e. a pure integrator to achieve a true integral action in our LQG controller. However, the extended model will not be stabilizable (we can't affect the noise model integrator state with the control signal) with a pure integrator. To compute an LQG controller we require that our system model is stabilizable. If we instead use $H = \frac{1}{s+\delta}$, the system model is stabilizable since the noise model now is asymptotically stable.

b. A state space representation of the noise model is:

$$\begin{aligned}\dot{x}_w(t) &= -\delta x_w(t) + n(t)\\ w(t) &= x_w(t)\end{aligned}$$

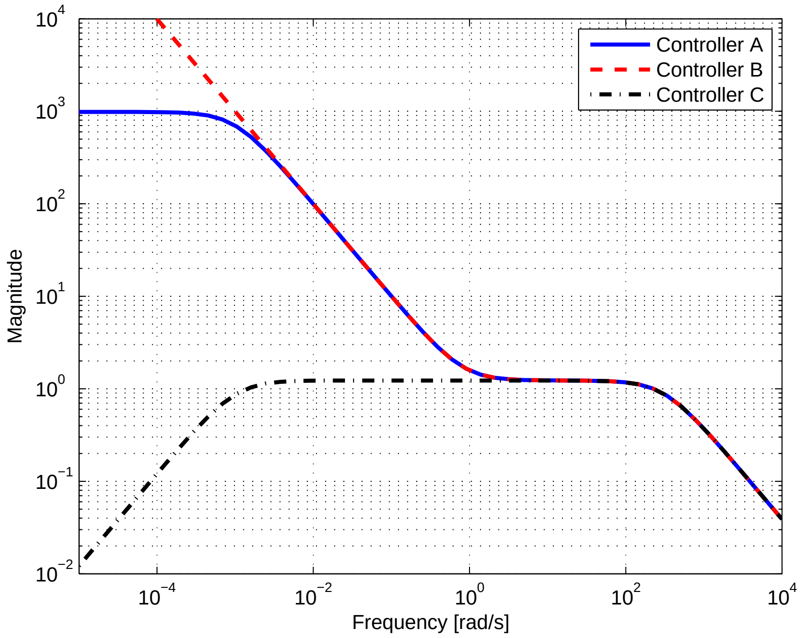

Figure 4 Bode diagram of three possible candidates for the LQG controller in Problem 6.

Inserting this into the initial model yields the extended model:

$$\begin{aligned}
\begin{bmatrix} \dot{x}(t) \\ \dot{x}_w(t) \end{bmatrix} &= \begin{bmatrix} A & 0 \\ 0 & -\delta \end{bmatrix} \begin{bmatrix} x(t) \\ x_w(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} v_1(t) \\ n(t) \end{bmatrix} \\
z(t) &= \begin{bmatrix} C & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ x_w(t) \end{bmatrix} \\
y(t) &= \begin{bmatrix} C & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ x_w(t) \end{bmatrix} + v_2(t)
\end{aligned}$$

That is

$$A_e = \begin{bmatrix} A & 0 \\ 0 & \delta \end{bmatrix} \quad B_e = \begin{bmatrix} B \\ 0 \end{bmatrix} \quad C_e = \begin{bmatrix} C & 1 \end{bmatrix}$$

c. The three controllers only differ for low frequencies. With the addition of the noise model we expect an LQG controller which has a high gain for low frequencies, and thus controller C can't be an LQG controller based on the extended model.

With true integral action the controller's static gain would be infinite. However, we can't achieve true integral action with this method, although we can get arbitrarily close by letting δ approach zero. Thus the controller gain has to level out eventually for low frequencies, at some large (but not infinite) gain. Since controller B does not level out, only controller A can be a possible LQG controller based on the extended model.

7. Consider the closed-loop system in Figure 5, where P_0 is a SISO system. A controller should be designed to attenuate the effect of input disturbances d on the process output x, while keeping the effect of measurement noise n on the control signal u to a minimum.

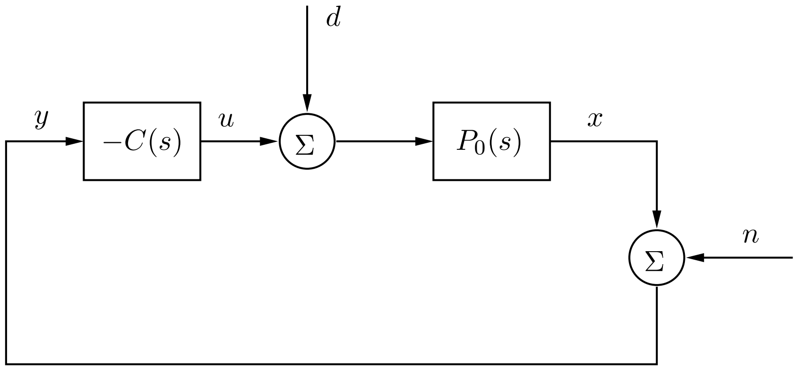

Figure 5 Block diagram of the closed-loop system in Problem 7.

a. Define $z = [\,x \quad u\,]^T$ and $w = [\,d \quad n\,]^T$, and rewrite the system in Figure 5 to the more general form in Figure 6. Provide the expressions for P_{zw}, P_{zu}, P_{yw} and P_{yu}. (1 p)

b. Determine the closed-loop system from w to z in terms of P_0 and C. (1 p)

c. Now assume $P_0(s) = \frac{1}{s+2}$. Two controllers, $C_1(s) = \frac{1}{s(s+3)}$ and $C_2(s) = 2$, have been found to achieve stable closed-loop systems. Determine the Q-parameterization for each controller, where the parameter $Q(s)$ should be a stable transfer function. The resulting closed-loop system should be linear in $Q(s)$. (1 p)

d. The response of the closed-loop system is tested for a step input in d and an impulse in n for each controller. Also, the $\mathcal{L}_2$-gain of the closed-loop system is computed for each case. The results are:

Evaluation	C_1	C_2
Maximum value of $x(t)$ after unit step in d.	0.44	0.25
Minimum value of $u(t)$ after unit impulse in n.	−0.14	−2.0
$\mathcal{L}_2$-gain of closed-loop transfer function from n to u.	1.0	0.5

Determine a controller which achieves $x(t) \leq 0.4$ after a unit step in d and $u(t) \geq -0.8$ after an impulse in n. Additionally, the maximum allowable $\mathcal{L}_2$-gain of the transfer function from n to u is 0.95.

Note: You can express your controller in terms of the Q-parameters $Q_1(s)$ and $Q_2(s)$ for the controllers $C_1(s)$ and $C_2(s)$ respectively. (1 p)

Solution

a. With the given definition of z and u, by analyzing the block diagram we obtain

$$P = \begin{bmatrix} P_{zw} & P_{zu} \\ P_{yw} & P_{yu} \end{bmatrix} = \begin{bmatrix} P_{xd} & P_{xn} & P_{xu} \\ P_{ud} & P_{un} & P_{uu} \\ P_{yd} & P_{yn} & P_{yu} \end{bmatrix} = \begin{bmatrix} P_0 & 0 & P_0 \\ 0 & 0 & 1 \\ P_0 & 1 & P_0 \end{bmatrix}$$

where

$$P_{zw} = \begin{bmatrix} P_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_{zu} = \begin{bmatrix} P_0 \\ 1 \end{bmatrix}, \quad P_{yw} = [\,P_0 \quad 1\,], \quad P_{yu} = P_0$$

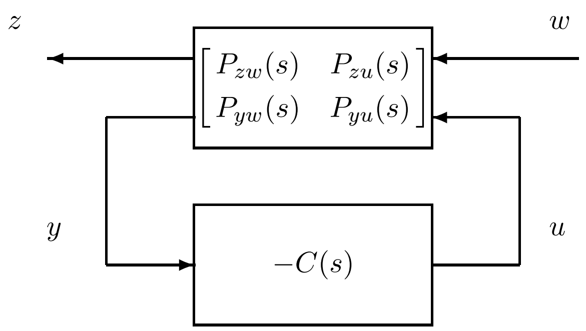

Figure 6 General closed-loop system in Problem 7.

b. The closed-loop system H from w to z is:

$$H = P_{zw} - P_{zu}C(1 + P_{yu}C)^{-1}P_{yw} = \begin{bmatrix} P_0 & 0 \\ 0 & 0 \end{bmatrix} - \frac{C}{1 + P_0 C} \begin{bmatrix} P_0 \\ 1 \end{bmatrix} \begin{bmatrix} P_0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{P_0}{1+P_0C} & -\frac{P_0C}{1+P_0C} \\ -\frac{P_0C}{1+P_0C} & -\frac{C}{1+P_0C} \end{bmatrix}$$

c. Every controller $C(s)$ which stabilizes the system can be parameterized with a stable transfer function $Q(s)$ as:

$$C(s) = \frac{Q(s)}{1 - Q(s)P_0(s)}$$

We can thus calculate $Q(s)$ for a given controller $C(s)$:

$$Q(s) = \frac{C(s)}{1 + P_0(s)C(s)}$$

This parametrization results in a closed-loop system which is linear in $Q(s)$. For C_1 we get:

$$Q_1(s) = \frac{C_1(s)}{1 + P_0(s)C_1(s)} = \frac{s + 2}{s^3 + 5s^2 + 6s + 1}$$

and C_2:

$$Q_2(s) = \frac{C_2(s)}{1 + P_0(s)C_2(s)} = \frac{2(s + 2)}{s + 4}$$

d. All of the specifications are convex in $Q(s)$, and either $Q_1(s)$ or $Q_2(s)$ (parameterizing $C_1(s)$ and $C_2(s)$ respectively) fulfills either of the specifications. Thus we can search for a convex combination of $Q_1(s)$ and $Q_2(s)$ which fulfills all of the specifications. One such convex combination is $Q_3(s) = 0.7Q_1(s) + 0.3Q_2(s)$ since it will result in:

$$\max x(t) \leq 0.7 \cdot 0.44 + 0.3 \cdot 0.25 = 0.383 \leq 0.4$$
$$\min u(t) \geq 0.7 \cdot (-0.14) + 0.3 \cdot (-2.0) = -0.698 \geq -0.8$$
$$\left\| \frac{P_0 C_3}{1 + P_0 C_3} \right\|_\infty \leq 0.7 \cdot 1.0 + 0.7 \cdot 0.5 = 0.85 \leq 0.95$$

With this $Q_3(s)$ we have the resulting controller

$$C_3(s) = \frac{Q_3(s)}{1 - Q_3(s)P_0} = \frac{0.7Q_1 + 0.3Q_2}{1 - (0.7Q_1 + 0.3Q_2)P_0}$$