Solution to Exam in FRTN10 Multivariable Control 2016-10-25

1 a. The 1×1 minors of $G(s)$ are

$$\frac{s}{s+1}, \quad \frac{-2(s+1)}{s+2}, \quad 3, \quad \frac{1}{s+2}$$

and the 2×2 determinant is

$$\frac{s}{s+1} \cdot \frac{1}{s+2} - 3 \cdot \frac{-2(s+1)}{s+2} = \frac{6s^2 + 13s + 6}{(s+1)(s+2)}$$

The least common denominator all subdeterminants is $(s+1)(s+2)$, thus the poles of $G(s)$ are -1 and -2, both with multiplicity 1.

b. The maximal minor is the 2×2 determinant $\dfrac{6s^2 + 13s + 6}{(s+1)(s+2)}$. The zeros are given by the roots of the numerator polynomial, which are

$$s = -\frac{13}{12} \pm \frac{5}{12}$$

giving the zeros $-3/2$ and $-2/3$. The zeros are not in the RHP and hence do not impose any fundamental limitations.

2 a. The controllability Gramian is the symmetric 2×2 matrix S_x that solves the Lyapunov equation $AS_x + S_xA^T + BB^T = 0$. Straightforward calculations give

$$S_x = \begin{pmatrix} 20 & 0 \\ 0 & 0.1 \end{pmatrix}$$

The system is controllables since the controllability Gramian is non-singular. The second state is much harder to control than the first one, the second eigenvalue of S_x being 200 times smaller than the first one.

b. We have

$$S_xO_x = \begin{pmatrix} 20 & 0 \\ 0 & 1 \end{pmatrix}$$

Since S_xO_x is diagonal, the squared Hankel singular values are immediately given by the diagonal elements, implying $\sigma_1 = \sqrt{20}$ and $\sigma_2 = 1$. You would keep the state corresponding to the larger singular value, i.e. x_1. The error bound is

$$\frac{||y - y_r||_2}{||u||_2} \leq 2\sigma_2 = 2$$

if we reduce the system to first order.

3 a. The static gain matrix is

$$G(0) = \begin{pmatrix} 12.8 & -18.9 \\ 6.6 & -19.4 \end{pmatrix}$$

from which the RGA in stationarity becomes

$$G(0) \,.\!*\, G(0)^{-T} = \begin{pmatrix} 2.0094 & -1.0094 \\ -1.0094 & 2.0094 \end{pmatrix}$$

From this we can conclude that we should couple $u_1 \leftrightarrow y_1$ and $u_2 \leftrightarrow y_2$. However, the RGA is far from an identity matrix, so there will be visible interation.

b. The matrices could for instance be chosen as $W_1 = I$ and $W_2 = G(0)^{-1}$.

c.
i. False! $\tilde{G}$ is only decoupled in stationarity, so all dynamic behavior will still give rise to cross-couplings. Also, the outputs of the closed loop system G is not necessarily decoupled just because the outputs of $\tilde{G}$ are. This we saw in for instance lab 2, where the tank levels were still coupled, but the sum and difference of them were decoupled.

ii. False! With the decoupler we do not change the physical process at all, just the control structure.

iii. False! Fundamental limitations cannot be removed by any controller structure.

iv. True. The decentralized structure means that we design SISO controllers, and the decoupler takes the MIMO cross-coupling in stationarity into account.

4 a. From the block diagram, setting $n = 0$, we see that

$$U = C(R - P(D + U))$$

Solving for U we obtain

$$U = \begin{pmatrix} (I + CP)^{-1}C & -(I + CP)^{-1}CP \end{pmatrix} \begin{pmatrix} R \\ D \end{pmatrix}$$

b. $C(s)$ must have 3 inputs and 2 outputs, r must be a vector of size 3 and d must be a vector of size 2.

c. No. The gain of each system is given by the maximum of the largest singular value. The gain of P is larger than 0.7 and the gain of C is larger than 3. The loop gain is hence larger than 1 and stability can not be asserted using the Small Gain Theorem.

5 a. The inverse of the maximum sensitivity, $1/M_s$, measures the minimum distance between the Nyquist curve and the critical point -1. Here, $M_s \approx 6$, implying that the robustness is poor. The guaranteed amplitude margin is only $A_m = \frac{M_s}{M_s - 1} = 1.2$.

The plant has a non-minimum-phase (NMP) zero in 2. The rule of thumb for unstable zeros then says that the bandwidth of the closed-loop system cannot be faster than 2 rad/s (and should not be faster than 1 rad/s to ensure $M_s \leq 2$).

b. The Maximum Modulus Theorem implies that the specification $\|W_s S\|_\infty \leq 1$ is impossible to fulfill if $|W_s(z)| > 1$, where z is the location of the NMP zero. We have

$$|W_s(2)| = \frac{2 + M_s \omega_0}{2M_s}$$

which is always greater than 1 if $\omega_0 \geq 2$.

With $M_s = 1.4$ it is impossible to fulfill the specification if

$$|W_s(2)| = \frac{2 + 1.4\omega_0}{2 \cdot 1.4} > 1 \quad \Rightarrow \quad \omega_0 > 0.57$$

6 a. We would ideally like to have a noise model which has an infinite amplification for static signals, i.e. a pure integrator to achieve a true integral action in our LQG controller. However, the extended model will not be stabilizable (we can't affect the noise model integrator state with the control signal) with a pure integrator. To compute an LQG controller we require that our system model is stabilizable. If we instead use $H = \frac{1}{s+\delta}$, the system model is stabilizable since the noise model now is asymptotically stable.

b. A state space representation of the noise model is:

$$\begin{aligned} \dot{x}_w(t) &= -\delta x_w(t) + n(t) \\ w(t) &= x_w(t) \end{aligned}$$

Inserting this into the initial model yields the extended model:

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{x}_w(t) \end{bmatrix} &= \begin{bmatrix} A & 0 \\ 0 & -\delta \end{bmatrix} \begin{bmatrix} x(t) \\ x_w(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} v_1(t) \\ n(t) \end{bmatrix} \\ z(t) &= \begin{bmatrix} C & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ x_w(t) \end{bmatrix} \\ y(t) &= \begin{bmatrix} C & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ x_w(t) \end{bmatrix} + v_2(t) \end{aligned}$$

That is

$$A_e = \begin{bmatrix} A & 0 \\ 0 & \delta \end{bmatrix} \quad B_e = \begin{bmatrix} B \\ 0 \end{bmatrix} \quad C_e = \begin{bmatrix} C & 1 \end{bmatrix}$$

c. The three controllers only differ for low frequencies. With the addition of the noise model we expect an LQG controller which has a high gain for low frequencies, and thus controller C can't be an LQG controller based on the extended model.

With true integral action the controller's static gain would be infinite. However, we can't achieve true integral action with this method, although we can get arbitrarily close by letting δ approach zero. Thus the controller gain has to level out eventually for low frequencies, at some large (but not infinite) gain. Since controller B does not level out, only controller A can be a possible LQG controller based on the extended model.

7 a. With the given definition of z and u, by analyzing the block diagram we obtain

$$P = \begin{bmatrix} P_{zw} & P_{zu} \\ P_{yw} & P_{yu} \end{bmatrix} = \begin{bmatrix} P_{xd} & P_{xn} & P_{xu} \\ P_{ud} & P_{un} & P_{uu} \\ P_{yd} & P_{yn} & P_{yu} \end{bmatrix} = \begin{bmatrix} P_0 & 0 & P_0 \\ 0 & 0 & 1 \\ P_0 & 1 & P_0 \end{bmatrix}$$

where

$$P_{zw} = \begin{bmatrix} P_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_{zu} = \begin{bmatrix} P_0 \\ 1 \end{bmatrix}, \quad P_{yw} = \begin{bmatrix} P_0 & 1 \end{bmatrix}, \quad P_{yu} = P_0$$

b. The closed-loop system H from w to z is:

$$H = P_{zw} - P_{zu}C(1 + P_{yu}C)^{-1}P_{yw} = \begin{bmatrix} P_0 & 0 \\ 0 & 0 \end{bmatrix} - \frac{C}{1 + P_0 C}\begin{bmatrix} P_0 \\ 1 \end{bmatrix}\begin{bmatrix} P_0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{P_0}{1+P_0C} & -\frac{P_0C}{1+P_0C} \\ -\frac{P_0C}{1+P_0C} & -\frac{C}{1+P_0C} \end{bmatrix}$$

c. Every controller $C(s)$ which stabilizes the system can be parameterized with a stable transfer function $Q(s)$ as:

$$C(s) = \frac{Q(s)}{1 - Q(s)P_0(s)}$$

We can thus calculate $Q(s)$ for a given controller $C(s)$:

$$Q(s) = \frac{C(s)}{1 + P_0(s)C(s)}$$

This parametrization results in a closed-loop system which is linear in $Q(s)$. For C_1 we get:

$$Q_1(s) = \frac{C_1(s)}{1 + P_0(s)C_1(s)} = \frac{s+2}{s^3 + 5s^2 + 6s + 1}$$

and C_2:

$$Q_2(s) = \frac{C_2(s)}{1 + P_0(s)C_2(s)} = \frac{2(s+2)}{s+4}$$

d. All of the specifications are convex in $Q(s)$, and either $Q_1(s)$ or $Q_2(s)$ (parameterizing $C_1(s)$ and $C_2(s)$ respectively) fulfills either of the specifications. Thus we can search for a convex combination of $Q_1(s)$ and $Q_2(s)$ which fulfills all of the specifications. One such convex combination is $Q_3(s) = 0.7Q_1(s) + 0.3Q_2(s)$ since it will result in:

$$\begin{aligned} \max x(t) &\leq 0.7 \cdot 0.44 + 0.3 \cdot 0.25 = 0.383 \leq 0.4 \\ \min u(t) &\geq 0.7 \cdot (-0.14) + 0.3 \cdot (-2.0) = -0.698 \geq -0.8 \\ \left\| \frac{P_0 C_3}{1 + P_0 C_3} \right\|_\infty &\leq 0.7 \cdot 1.0 + 0.7 \cdot 0.5 = 0.85 \leq 0.95 \end{aligned}$$

With this $Q_3(s)$ we have the resulting controller

$$C_3(s) = \frac{Q_3(s)}{1 - Q_3(s)P_0} = \frac{0.7Q_1 + 0.3Q_2}{1 - (0.7Q_1 + 0.3Q_2)P_0}$$