LUND UNIVERSITY

Department of
AUTOMATIC CONTROL

FRTN10 Multivariable Control

Exam 2015-01-09

Points and grades

All answers must include a clear motivation and a well-formulated answer. Answers may be given in English or Swedish. The total number of points is 25. The maximum number of points is specified for each subproblem.

Accepted aid

The textbook *Glad & Ljung*, standard mathematical tables like TEFYMA, an authorized "Formelsamling i Reglerteknik"/"Collection of Formulas" and a pocket calculator. Handouts of lecture notes and lecture slides are also allowed.

Results

The results will be reported via LADOK.

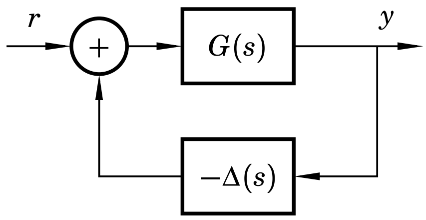

Figure 1 Setup for the system in problem 1.

1. A multivariable system is given by

$$G(s) = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & 0 \\ 0 & G_{32} \end{pmatrix}$$

a. Draw a block diagram for $G(s)$ showing the connections between each input and each output. (1 p)

b. The system is connected to an unknown function $\Delta(s)$ as described in Figure 1. Some plots for $G(s)$ are shown in Figure 2. What restrictions do you need to impose on the function $\Delta(s)$ to ensure that the closed-loop system will be stable, according to the Small Gain Theorem? (1 p)

Solution

a. The connections between inputs and outputs are shown in Figure 3.

b. Since the maximum value in the singular value plot is three, we know from the small gain theorem that the closed loop system will be stable if $||\Delta||_\infty < 1/3$.

2. A system is given by the state equations

$$\dot{x}(t) = \begin{pmatrix} -3 & 0 \\ 0 & -5 \end{pmatrix} x(t) + \begin{pmatrix} 3/2 & 0 & 3 \\ -1/2 & 1 & 0 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 1 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} u(t).$$

a. How many inputs, outputs and states does the system have? (1 p)

b. Find a transfer function for the system. (2 p)

c. What are the poles and zeros of the multivariable system? (2 p)

Solution

a. By looking at the dimensions of the matrices we can conclude that we have three inputs $u = (u_1\ u_2\ u_3)^T$, one output $y = y_1$ and two states $x = (x_1\ x_2)^T$.

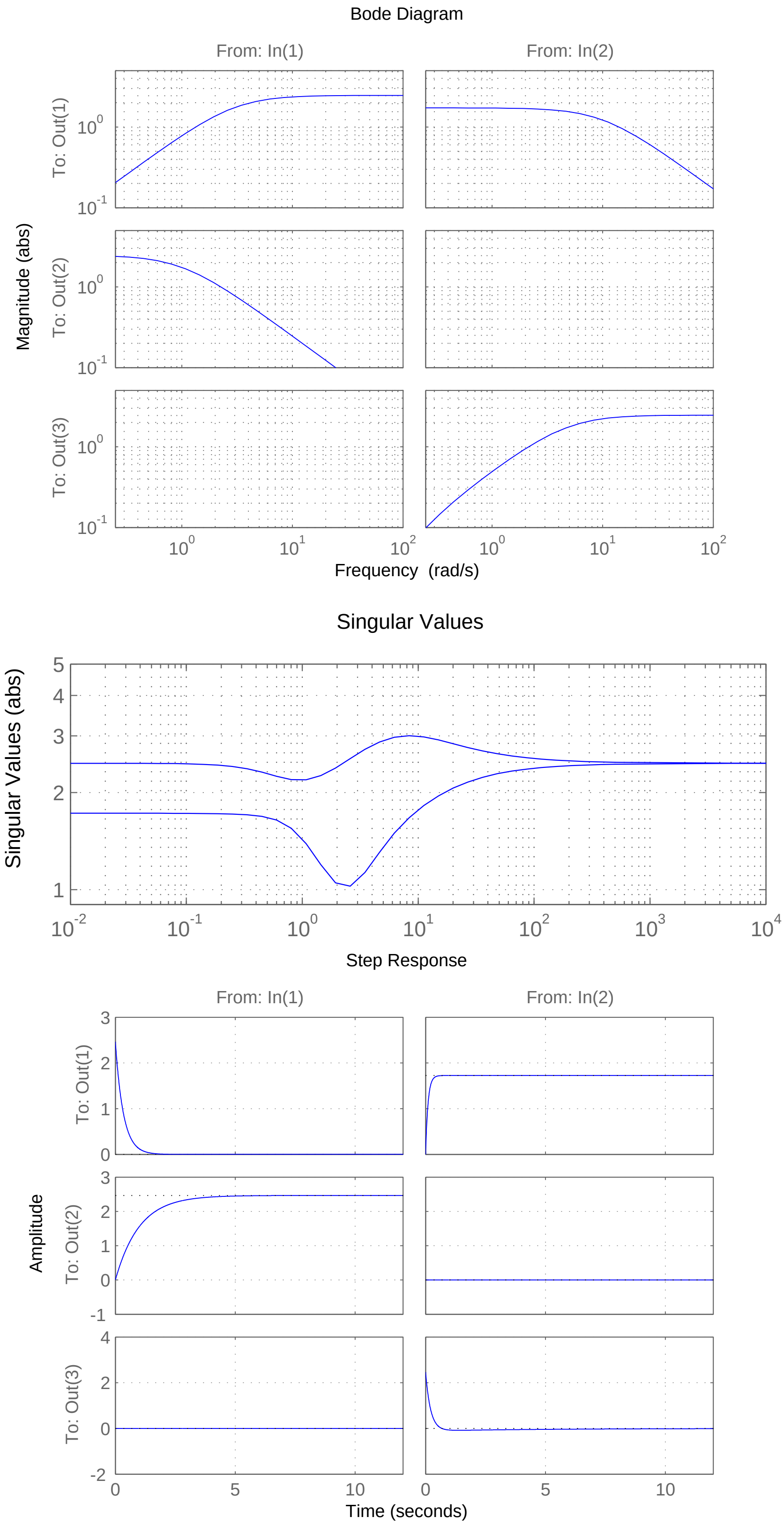

Figure 2 Some different plots for the system $G(s)$ in problem 1.

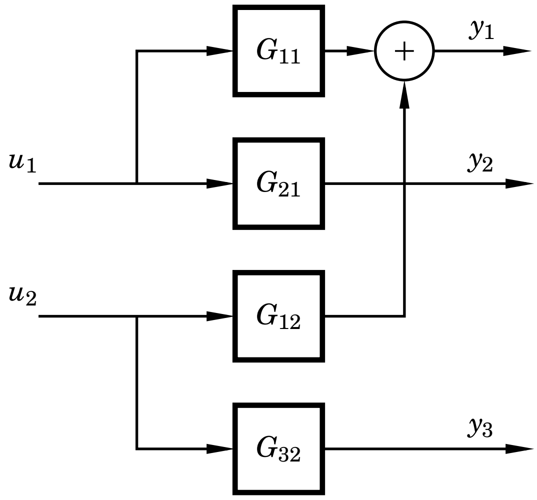

Figure 3 Connections between inputs and outputs in problem 1.

b. The transfer function of the system is given by $G(s) = C(sI - A)^{-1}B + D$.

$$
\begin{aligned}
G(s) &= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{s+3} & 0 \\ 0 & \frac{1}{s+5} \end{pmatrix} \begin{pmatrix} 3/2 & 0 & 3 \\ -1/2 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{s+3} & \frac{1}{s+5} \end{pmatrix} \begin{pmatrix} 3/2 & 0 & 3 \\ -1/2 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{3/2}{s+3} - \frac{1/2}{s+5} & \frac{1}{s+5} + 1 & \frac{3}{s+3} + 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{s+6}{(s+3)(s+5)} & \frac{s+6}{s+5} & \frac{s+6}{s+3} \end{pmatrix}
\end{aligned}
$$

c. The poles to the system are -3 and -5 as can be seen already in the state-space formulation. The system zeros are given by looking at the greatest common divisor of the numerators of the maximal minors of $G(s)$, normalized so that you have the pole polynomial in the denominator. The maximal minors are $\dfrac{s+6}{(s+3)(s+5)}$, $\dfrac{(s+6)(s+5)}{(s+3)(s+5)}$ and $\dfrac{(s+3)(s+6)}{(s+3)(s+5)}$ and they have the common divisor $s+6$ which means that we have a system zero in $s = -6$.

3. Your boss gives you what he claims are RGA matrices in stationarity for three different systems S_1, S_2 and S_3 that are all important to your firm. For each of the systems, check if the matrix is a valid RGA matrix, and describe if and how you would use the given information to design your controller structure.

$$
RGA(S_1) = \begin{pmatrix} 0.55 & 0.45 \\ 0.45 & 0.55 \end{pmatrix}
$$

$$
RGA(S_2) = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}
$$

$$
RGA(S_3) = \begin{pmatrix} 1.2 & 0.1 \\ 0.1 & 1.2 \end{pmatrix}
$$

(2 p)

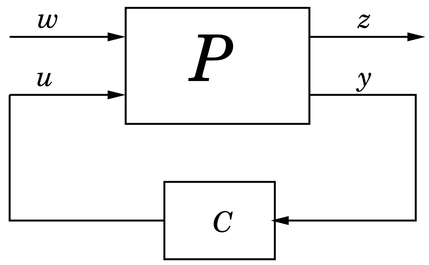

Figure 4 System description for problem 4.

Solution

For S_1 the best pairing is to have the first input controlling the first output. However, since the values are so alike it is probably not a good idea to use RGA pairing for this system. For S_2 you would instead like to control the first output with the second input. The matrix given for S_3 is clearly not an RGA matrix since the rows and columns do not sum to 1, therefore you wouldn't use that one to make any decisions about controller design at all.

4. You want to find a controller for your stable process P by using some convex optimization tool. The closed loop system is shown in Figure 4 where

$$z = \begin{pmatrix} e \\ u \end{pmatrix}, \quad w = \begin{pmatrix} r \\ d \\ n \end{pmatrix}, \quad y = \begin{pmatrix} r \\ x+n \end{pmatrix}$$

and e, u, r, d, n, x are all scalars. Your aim is to minimize the cost function

$$J = \int_0^\infty \alpha e^2(t) + \beta u^2(t) dt$$

under the restriction that the closed loop system should be stable. The closed loop has the Youla parametrization $H_{zw} = P_{zw} + P_{zu} Q P_{yw}$, where you need to find Q to satisfy your constraints and optimize your cost function.

a. For the following matrices, motivate which ones (if any) are feasible Q-matrices.

$$Q_1 = \frac{1}{s+1}, \qquad Q_2 = \begin{pmatrix} \frac{1}{s+1} & \frac{1}{s+3} \end{pmatrix}, \quad Q_3 = \begin{pmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s-3} \end{pmatrix},$$

$$Q_4 = \begin{pmatrix} \frac{2}{s+1} \\ \frac{1}{s+3} \end{pmatrix}, \quad Q_5 = \begin{pmatrix} \frac{1}{s-1} & \frac{1}{s+1} \end{pmatrix}, \quad Q_6 = \begin{pmatrix} \frac{1}{s+1} & 1 \\ \frac{1}{s+2} & \frac{1}{s+3} \end{pmatrix}.$$

(1 p)

b. You might want to add some additional constraints to your optimization problem. Which of the following constraints could you add and still have a convex optimization problem? Here $f_i(Q),\ i = 1, 2, 3$ denotes different affine functions of Q.

$$
\begin{aligned}
&I) \quad ||f_1(Q)||_\infty \le \alpha_1 \\
&II) \quad ||f_2(Q)||_2 \ge \alpha_2 \\
&III) \quad f_1(Q) + f_2(Q) = \beta_1 \\
&IV) \quad ||f_1(Q) + f_3(Q)|| = \beta_2 \\
&V) \quad \log u = \beta_3
\end{aligned}
$$

(1 p)

Solution

a. From the equation for H_{zw} and knowing the dimensions of u and y we can conclude that the dimension of Q must be [1x2], that excludes all matrices except Q_2 and Q_5. Since the closed loop system should be stable we also know that Q must be stable, hence the only feasible matrix in this set of Q-matrices is Q_2.

b. A convex optimization problem can have constraints on the form

$$
\begin{aligned}
g_i(x) &\le 0 \\
h_i(x) &= 0
\end{aligned}
$$

where the functions $g_i(x)$ must be convex, and the functions $h_i(x)$ must be affine.
I) OKAY since norms are convex.
II) NOT OKAY since we can not have constraints on the form convex $\ge$ something.
III) OKAY since the sum of two affine functions is still affine.
IV) NOT OKAY since the norm is not an affine function and can not be in an equality constraint.
V) NOT OKAY since log u is a concave (not affine) function and can not be in an equality constraint.

5. Consider the process

$$P(s) = \frac{1}{s - \alpha}$$

where $\alpha > 0$. You want to stabilize the system by using a controller connected as shown in figure 5.

a. Your lazy friend wants to use the controller $C(s) = (s - \alpha)$ to get rid of the unstable pole. Give two reasons to why this is not a good idea. (1 p)

b. Find the parameters of a PI controller on the form

$$C(s) = \frac{K(sT_i + 1)}{sT_i}$$

that stabilizes the system. (1 p)

c. In addition to making the system stable you would also like to reduce the sensitivity to disturbances and measurement noise. Therefore you would like the transfer function from noise n to process output z, as well as the transfer function from the disturbance v to z to always have absolute values smaller than 1/2. Will this be possible? Motivate your answer. (1 p)

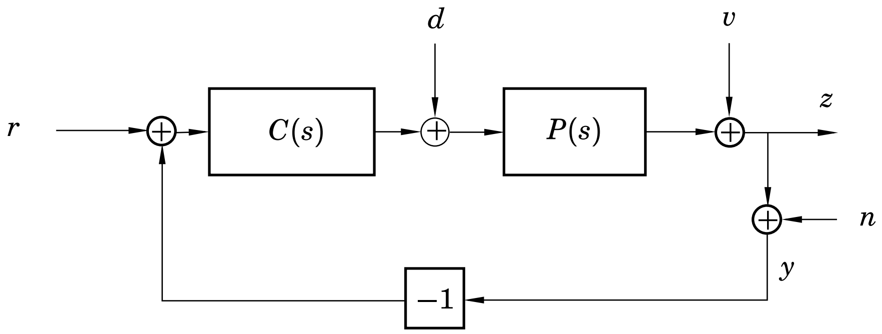

Figure 5 System in problem 5

Solution

a. The controller is not proper and can not be realized. Pole-zero cancellations of unstable poles are never a good idea. It might look good on the transfer function from r to z but will for example still be unstable for the transfer function from d to z.

b. The closed loop system is given by $G_{cl} = \dfrac{PC}{1+PC}$. Since $PC = \dfrac{K(sT_1+1)}{(s-\alpha)sT_i}$ the characteristic polynomial will be

$$s^2T_i + (K-\alpha)T_is + K = 0 \Rightarrow s^2 + (K-\alpha)s + K/T_i = 0.$$

This gives the stability conditions $K > \alpha$ and $T_i > 0$.

c. No, it will not be possible. The transfer function from n to z is

$$G_{zn} = \frac{-PC}{1+PC} = -T$$

and the transfer function from v to z is

$$G_{zv} = \frac{1}{1+PC} = S$$

and as we know the sum of $S+T=1$. This implies that they can not both have an absolute value below 1/2 at the same frequency since $S+T=1 \Rightarrow |S+T| = 1$ and $|S+T| \leq |S| + |T|$.

6. A system is given by

$$\dot{x}(t) = -2x(t) + u(t) + v_1(t)$$
$$y(t) = x(t) + w(t)$$

where $w(t)$ is low-pass filtered white noise,

$$w(t) = \frac{1}{s+1}v_2(t).$$

(Note that this is not very common since measurement noise is usually modeled to be mainly high frequent.) The intensity of v_2 is $R_2 = 5$, while $v_1(t)$ is white noise with intensity $R_1 = 1$.

a. Extend the state-space model to include the noise dynamics. (2 p)

b. Find the control law that minimizes the cost function

$$\int_0^\infty 5x(t)^2 + u(t)^2 dt.$$

(2 p)

Solution

a. $w(t) = \frac{1}{s+1}v_2(t)$ gives $\dot{w} + w = v_2$. By extending the state vector with the new state w to $x_e(t) = (x(t), w(t))^T$ we get

$$\begin{aligned} \dot{x}_e &= \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} x_e + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u + \begin{pmatrix} 1 \\ 0 \end{pmatrix} v_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v_2. \\ y &= \begin{pmatrix} 1 & 1 \end{pmatrix} x_e \end{aligned}$$

b. The minimizing control law is given by $u = -Lx$ where $L = Q_2^{-1}B^TS$. S is given as the symmetric, positive semi-definite solution to the Riccati equation

$$A^TS + SA + Q_1 - SBQ_2^{-1}B^TS = 0.$$

Considering the extended system from a) the cost function tells us that $Q_1 = \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}$ and $Q_2 = 1$. Calculations give $S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $L = \begin{pmatrix} 1 & 0 \end{pmatrix}$. This gives the control law $u(t) = -x(t)$.

7. A margin plot of a process is given in figure 6. Your task is to add controllers and/or filters in order to satisfy the following specifications

- There should be no stationary error for a step change in the reference
- The cross-over frequency should stay (approximately) the same
- You should have a phase margin of at least 35°.

To your help you have the controllers/filters $F_1 - F_8$ shown in Figure 7.

a. Combine the Bode diagram of each of the filters plotted in solid lines (i.e. F_1, F_3, F_5, F_7) with its corresponding transfer function $C_1 - C_6$.

$$C_1(s) = \frac{s/10+1}{s+1}, \quad C_2(s) = 0.5\frac{s+1}{s}, \quad C_3(s) = \frac{s/0.05+1}{s+1},$$
$$C_4(s) = \frac{s}{s+1}, \quad C_5(s) = \frac{1}{s+1}, \quad C_6(s) = \frac{1}{(s+1)^2}.$$

(2 p)

b. Find a combination of the filters $F_1 - F_8$ that satisfies the constraints on your process. Note that you can choose from both the filters shown in solid lines and dashed lines, however, don't use more filters than necessary. Be sure to motivate that all constraints are satisfied. (2 p)

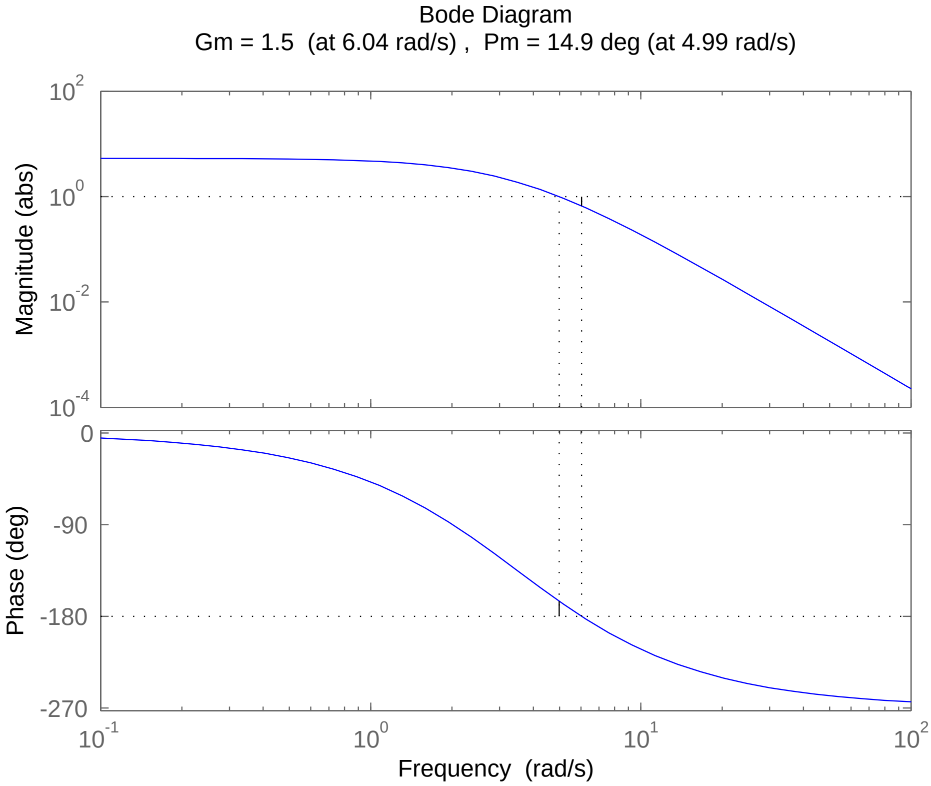

Figure 6 Margin plot of the process in problem 7

Solution

a. The solid line filters are given by the following transfer functions

$$F_1 - C_2$$
$$F_3 - C_1$$
$$F_5 - C_5$$
$$F_7 - C_3$$

b. By adding the filters F_2 and F_8 all the specifications will be satisfied. Motivation: To remove the stationary errors we need an integrator. F_1 and F_2 are PI controllers, both of them will decrease the gain of the wanted cross-over frequency ($\omega_c = 5$) by $1/2$. F_1 will decrease the phase at the wanted cross-over frequency more than F_2 so let's choose F_2. We also need to lift up the phase at $\omega_c = 5$, to do this we use a lead filter. The lead filters are given by F_7 and F_8. To get enough phase lift we pick F_8 which adds approximately $50°$ to the phase at $\omega_c = 5$. $|F_8(i\omega_c)| = 2$ so together with the PI controller that we added the gains will make sure that the cross-over frequency is not changed.

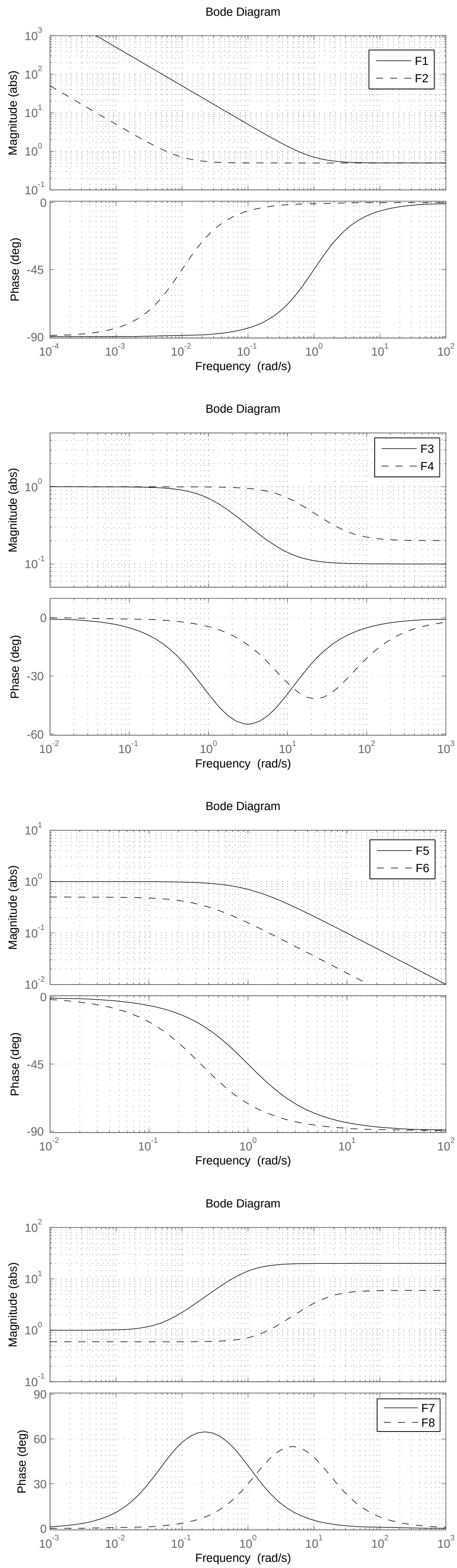

Figure 7 The available filters in problem 7.

8. A third-order system

$$\dot{x} = \begin{pmatrix} -1/2 & -16/17 & -2/9 \\ -4/17 & -2 & -5/3 \\ -8/9 & -16/3 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 16 & 2 \end{pmatrix} x$$

has the observability gramian

$$O_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

and controllability gramian

$$S_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 32 \end{pmatrix}.$$

a. Is this a balanced realization of the system? Motivate. (1 p)

b. You want to make a balanced truncation to get a first-order approximation of the system. Find out which states you would like to remove. (2 p)

Solution

a. No, a balanced system has $O_x = S_x$ which is not the case here.

b. By making a transformation $\xi = Tx$ I can find a balanced realization. With this transformation I have $S_\xi = TS_xT^T$ and $O_\xi = T^{-T}O_xT^{-1}$, so to have a balanced representation I need

$$S_\xi = O_\xi \Leftrightarrow TS_xT^T = T^{-T}O_xT^{-1}.$$

Since the gramians in this case are diagonal, the transformation matrix is also chosen diagonal. Then $T = T^T$ and we have that $TTS_xTT = O_x$. Since the matrices are all diagonal this gives the equations

$$\begin{cases} t_1^4 s_1 = o_1 \\ t_2^4 s_2 = o_2 \\ t_3^4 s_3 = o_3 \end{cases} \Rightarrow \begin{cases} t_1^4 = 1/1 \\ t_2^4 = 64/4 \\ t_3^4 = 2/32 \end{cases} \Rightarrow \begin{cases} t_1 = 1 \\ t_2 = 2 \\ t_3 = 1/2 \end{cases}.$$

The gramians are then

$$S_\xi = O_\xi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

As can be seen by looking at the gramians the state I would like to keep is the ξ_2 since it has the most influence. (Since T is diagonal this will also correspond to the second state in the original representation x_2.) So the states I would remove are ξ_1 and ξ_3.