1.

a. Start by determining the transfer function matrix of the system

$$G(s) = C(sI - A)^{-1}B + D = \begin{pmatrix} 10 & c \\ 10 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{(s+10)} & 0 \\ 0 & \frac{1}{(s+1)} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{10}{(s+10)} & \frac{c}{(s+1)} \\ \frac{10}{(s+10)} & 1 \end{pmatrix}$$

The zero polynomial is the numerator of $\det G(s)$ when the denominator is the pole polynomial. We have that

$$\det G(s) = 10\frac{(s+1-c)}{(s+10)(s+1)}.$$

Therefore the zero is located in $s = c - 1$ and thus we have a right half plane zero if and only if $c > 1$.

b. The observability matrix is

$$W_o = \begin{pmatrix} 10 & c \\ 10 & 0 \\ -100 & -c \\ -100 & 0 \end{pmatrix}$$

which has full rank, i.e., 2, for $c \neq 0$. Thus, we have observability only when $c \neq 0$.

c. The controllability gramian is determined by solving

$$AS + SA^T + BB^T = 0$$

which gives the following equations, when using $S = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}$,

$$\begin{aligned} -20s_1 + 1 &= 0 \\ -11s_2 &= 0 \\ -2s_3 + 1 &= 0 \end{aligned}$$

which gives

$$S = \begin{pmatrix} 0.05 & 0 \\ 0 & 0.5 \end{pmatrix}$$

2.

a. From lecture 6, the transfer matrix $\sum_{i=1}^{n} \frac{C_i B_i}{s-p_i} + D$ has the realization

$$\dot{x}(t) = \begin{bmatrix} p_1 I & & 0 \\ & \ddots & \\ 0 & & p_n I \end{bmatrix} x(t) + \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} u(t)$$

$$y(t) = [\, C_1 \quad \dots \quad C_n \,]\, x(t) + Du(t)$$

Hence

$$G(s) = \begin{pmatrix} \frac{2}{s^2+3s+2} & \frac{1}{s+1} \end{pmatrix} = \begin{pmatrix} \frac{2}{s+1} - \frac{2}{s+2} & \frac{1}{s+1} \end{pmatrix}$$
$$= \frac{1}{s+1} \underbrace{\begin{pmatrix} 2 & 1 \end{pmatrix}}_{B_1} + \frac{1}{s+2} \underbrace{\begin{pmatrix} -2 & 0 \end{pmatrix}}_{B_2}$$

with $p_1 = -1$, $p_2 = -2$ and $C_1 = C_2 = 1$ gives

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} x + \begin{pmatrix} 2 & 1 \\ -2 & 0 \end{pmatrix} u = Ax + Bu$$
$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} x = Cx$$

b.

$$\Phi_d(\omega) = \left| \frac{1}{(\omega^2+1)(\omega^2+4)} \right|$$
$$= \left| \frac{1}{(i\omega+1)(-i\omega+1)(i\omega+2)(-i\omega+2)} \right|$$
$$= \left| \frac{1}{(i\omega+1)(i\omega+2)} \cdot \frac{1}{(-i\omega+1)(-i\omega+2)} \right|$$
$$= |H(i\omega)H(-i\omega)| \Rightarrow$$
$$H(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s^2+3s+2}$$

c. The Laplace transform of the disturbance signal d can be written as $D(s) = \frac{1}{s^2+3s+2} W(s)$ where w is white noise. By introducing the states $x_{d_1} = d$ and $x_{d_2} = \dot{d}$ we can write the noise model in state space form as

$$\dot{x}_d = \begin{pmatrix} \dot{x}_{d_1} \\ \dot{x}_{d_2} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}}_{A_d} x_d + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B_d} w$$

Extending the state space model from **a.** with the new noise states and by rewriting the B matrix as $B = \begin{pmatrix} B_1 & B_2 \end{pmatrix}$ gives

$$\begin{pmatrix} \dot{x} \\ \dot{x}_d \end{pmatrix} = \begin{pmatrix} A & [\, B_2 \quad 0_{2\times 1} \,] \\ 0_{2\times 2} & A_d \end{pmatrix} \begin{pmatrix} x \\ x_d \end{pmatrix} + \begin{pmatrix} B_1 & 0_{2\times 1} \\ 0_{2\times 1} & B_d \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix}$$

3.

a. The small gain theorem guarantees stability if

$$||\Delta||_\infty \cdot ||G_{un}||_\infty < 1,$$

where G_{un} is the transfer function from the output of Δ to the control signal (which is also the input of Δ). Note that the two transfer function must be input-output stable. We have that

$$G_{un} = \frac{-KW}{1+KP} = \frac{-sK}{s+3+3K}$$

and is input-output stable if $K > -1$. We see that $\|G_{un}\|_\infty \leq K$ and $\|\Delta\|_\infty < 1$, so any K satisfying $-1 \leq K \leq 1$ is ok.

b. The given weighting factor $W(s)$ tends to 0 for low frequencies since

$$\lim_{\omega \to 0} \frac{\omega^2}{\omega^2 + 3^2} = 0$$

This means that also $\Delta(s)W(s)$ tends to 0 since $\|\Delta\|_\infty < 1$. Hence, for low frequencies there is no uncertainty in the process model. Looking at the two plots we see that the left plot shows uncertainties at low frequencies while the right one does not, the dotted lines tend to the process model. Thus, plot B illustrates the uncertainty of the process with given $W(s)$.

4.

a. Given that

$$Q(s) = \frac{G^{-1}(s)}{(\lambda s+1)^2} = \frac{(s+1)(100s+1)}{(\lambda s+1)^2},$$

the controller can be derived using the formula

$$C(s) = \frac{Q(s)}{1 - Q(s)G(s)} = \frac{(s+1)(100s+1)}{\lambda s(\lambda s+2)}.$$

The complementary sensitivity function is given by (see the book)

$$T(s) = \frac{G(s)C(s)}{1+G(s)C(s)} = G(s)Q(s) = \frac{1}{(\lambda s+1)^2}$$

The slowest pole in the process is $-1/100$, hence we should choose $\lambda = 1$ which gives -1 as slowest pole in the closed loop system. The corresponding controller becomes

$$C(s) = \frac{(s+1)(100s+1)}{s(s+2)}$$

b. The closed loop transfer function from d to y is

$$\frac{G(s)}{1+G(s)C(s)} = \frac{\lambda s(\lambda s+2)}{(\lambda s+1)^2(s+1)(100s+1)}$$

and here the slow process pole will be visible no matter how small λ is made. The reason why this does not show in the transfer function from r to y is that the controller cancels out the slow pole.

5.

a. The Riccati equation

$$0 = Q_1 + A^T S + SA - (SB + Q_{12})Q_2^{-1}(SB + Q_{12})^T$$

with $A = -1$, $B = 1$, $Q_1 = 2$, $Q_{12} = 0$, $Q_2 = 1$ becomes

$$0 = 2 - 2S - S^2 = 3 - (S+1)^2$$

which gives the positive semidefinite solution

$$S = -1 + \sqrt{3}$$

and $L = S$. Hence the optimal state feedback is

$$u = (1 - \sqrt{3})x$$

b. For the Kalman filter we have the Riccati equation

$$0 = R_1 + AP + PA^T - (PC^T + R_{12})R_2^{-1}(PC^T + R_{12})^T$$

and $C = R_1 = R_2 = 1$, $R_{12} = 0$ gives

$$0 = 1 - 2P - P^2 = 2 - (P+1)^2$$

with the solution $P = -1+\sqrt{2}$ and Kalman filter gain $K = PC^T = -1+\sqrt{2}$. Thus, the controller is

$$\begin{aligned}\dot{\hat{x}}(t) &= -\hat{x}(t) + u(t) + (-1+\sqrt{2})(y(t) - C\hat{x}(t))\\ u(t) &= (1-\sqrt{3})x(t)\end{aligned}$$

c. For the system in **a.** the poles are given by the eigenvalues of $A - BL$. In this case there will be a single pole in $s = -\sqrt{3}$. The system in **b.** has an additional pole from the observer given by the eigenvalues of $A - KC$. This pole is located in $s = -\sqrt{2}$.

d. Minimization of

$$\mathbf{E}\left(y(t)^2 + \psi u(t)^2\right) = \psi \mathbf{E}\left(\frac{1}{\psi}y(t)^2 + u(t)^2\right)$$

is the same as minimization of

$$\mathbf{E}\left(\frac{1}{\psi}y(t)^2 + u(t)^2\right)$$

Hence, using $\psi = \dfrac{1}{2}$ will give the same control law as before.

6.

a. With bold face indicating Laplace transform, we have

$$
\begin{pmatrix} \mathbf{z} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{y} \\ \bar{\mathbf{u}} \\ \hline \mathbf{y} \end{pmatrix} = \begin{pmatrix} W_2 w_2 + P_0(W_1\mathbf{w}_1 + \mathbf{u}) \\ W_1\mathbf{w}_1 + \mathbf{u} \\ \hline W_2\mathbf{w}_2 + P_0(W_1\mathbf{w}_1 + \mathbf{u}) \end{pmatrix}
$$

$$
= \left(\begin{array}{cc|c} P_0 W_1 & W_2 & P_0 \\ W_1 & 0 & 1 \\ \hline P_0 W_1 & W_2 & P_0 \end{array}\right) \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \hline \mathbf{u} \end{pmatrix} = \begin{pmatrix} P_{zw}(s) & P_{zu}(s) \\ P_{yw}(s) & P_{yu}(s) \end{pmatrix} \begin{pmatrix} \mathbf{w} \\ \mathbf{u} \end{pmatrix},
$$

such that P_{zw} is a 2×2 matrix, P_{zu} is 2×1, P_{yw} 1×2 and P_{yu} 1×1.

b. The closed loop transfer function matrix H_{zw_2} from $\mathbf{w_2}$ to $\mathbf{z}$ can be determined using that

$$
H_{zw} = P_{zw} - P_{zu} C (1 + P_{yu} C)^{-1} P_{yw},
$$

which is the transfer function from $\mathbf{w}$ to $\mathbf{y}$. We have that

$$
H_{zw} = \begin{pmatrix} P_0 W_1 & W_2 \\ W_1 & 0 \end{pmatrix} - \begin{pmatrix} P_0 \\ 1 \end{pmatrix} \frac{C}{1 + P_0 C} \begin{pmatrix} P_0 W_1 & W_2 \end{pmatrix}
$$

$$
= \begin{pmatrix} P_0 W_1 & W_2 \\ W_1 & 0 \end{pmatrix} - \frac{C}{1 + P_0 C} \begin{pmatrix} P_0^2 W_1 & P_0 W_2 \\ P_0 W_1 & W_2 \end{pmatrix} = \begin{pmatrix} \dfrac{P_0 W_1}{1 + P_0 C} & \dfrac{W_2}{1 + P_0 C} \\ \dfrac{W_1}{1 + P_0 C} & \dfrac{-C W_2}{1 + P_0 C} \end{pmatrix}.
$$

Then, H_{zw_2} is the second column of H_{zw}, i.e.,

$$
H_{zw_2} = H_{zw} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \dfrac{W_2}{1 + P_0 C} \\ \dfrac{-C W_2}{1 + P_0 C} \end{pmatrix}.
$$

c. Since we want the sensitivity $S(s)$ to be small for low frequencies (as v is a low frequency disturbance), we can for instance choose

$$
W_2(s) = K \frac{sT + 1}{s}
$$

which has high gain at low frequencies. K changes the weight over all frequencies while T determines the cut-off frequency.