1. Since $S = 0$ and $S + T = 1 \Rightarrow T = 1$. The transfer function from measurement noise to process output also equals $-T(s)$. Since measurement often has high-frequency contents, we would like $T(s)$ to be small for high frequencies. Also, $T(s)$ maps relative errors in the model to the output. Since models typically are inaccurate for high frequencies, we would again like $T(s)$ to be small for high frequencies.

2. The cost function to be minimized is equivalent to the cost function

$$J = \int_0^\infty 10x^T C^T C x + u^2 \, dt = \int_0^\infty x^T \begin{pmatrix} 10 & 10 \\ 10 & 10 \end{pmatrix} x + u^2 \, dt.$$

The corresponding Riccati equation then becomes

$$A^T S + SA + Q_1 - SBB^T S = 0$$

with an S on the form

$$S = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}.$$

Calculations yield that a positive definite solution is given by

$$\begin{aligned} s_1 &= \sqrt{10}\sqrt{2\sqrt{10} + 10} - 10 \approx 2.77 \\ s_2 &= \sqrt{10} \approx 3.16 \\ s_3 &= \sqrt{2\sqrt{10} + 10} \approx 4.04 \end{aligned}$$

The optimal feedback vector is then given by

$$L = B^T S = \begin{pmatrix} 3.16 & 4.04 \end{pmatrix}$$

l_r is then calculated as

$$l_r = (C(BL - A)^{-1}B)^{-1} = 3.16$$

3. It is obvious from the Bode diagram that the process has an integrator, making the closed-loop transfer function from reference to output have unit static gain. However, the transfer function from input load disturbance to output, $P(s)C(s)$, will also have non-zero static gain, meaning that there will be a static error. As for the second part, there are a number problems. First, the phase margin for a P-controller at the desired cutoff frequency is around 10 degrees, which is really bad from a robustness perspective. Since a PI-controller never can give a net gain of phase, we cannot expect it to work by itself in this case. A potential fix for this is to combine the PI-controller with a lead filter in order to increase the phase around the cutoff frequency. The next problem is the magnitute of the resonance peak, which will spoil any effort to adjust the cutoff frequency using the system gain. We can fix this by adding lag filters or lowpass filters that pushed down the gain of the resonance peak and frequencies higher than the cutoff frequency, hopefully without affecting the area around the cutoff frequency too much.

4.

a. The disturbance v can simply be seen as inputs for both states and w is added to the output, the model is therefore

$$\begin{aligned}\dot{x} &= \begin{bmatrix}-1 & 0\\ 1 & -2\end{bmatrix}x + \begin{bmatrix}1\\0\end{bmatrix}u + \begin{bmatrix}1\\1\end{bmatrix}v\\ y &= \begin{bmatrix}0 & 1\end{bmatrix}x + w.\end{aligned}$$

b. Factorization results in:

$$\Phi_v(\omega) = \frac{(\frac{\omega}{\omega_0})^2}{\left((\frac{\omega}{\omega_0})^2+1\right)^2} = \underbrace{\frac{\frac{-i\omega}{\omega_0}}{\left(\frac{-i\omega}{\omega_0}+1\right)^2}}_{H^*(i\omega)} \frac{\frac{i\omega}{\omega_0}}{\left(\frac{i\omega}{\omega_0}+1\right)^2} = H^*(i\omega)H(i\omega),$$

which gives $H(s) = \frac{s/\omega_0}{(s/\omega_0+1)^2} = \frac{s\omega_0}{s^2+2s\omega_0+\omega_0^2}$.

c. The observable form (for example) can be found in the table of formulas and results in

$$\begin{aligned}\begin{bmatrix}\dot{x}_3\\ \dot{x}_4\end{bmatrix} &= \begin{bmatrix}-2\omega_0 & 1\\ -\omega_0^2 & 0\end{bmatrix}\begin{bmatrix}x_3\\ x_4\end{bmatrix} + \begin{bmatrix}\omega_0\\ 0\end{bmatrix}n\\ v &= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_3\\ x_4\end{bmatrix}\end{aligned}$$

so the full system can be written

$$\begin{aligned}\dot{x} &= \underbrace{\begin{bmatrix}-1 & 0 & 1 & 0\\ 1 & -2 & 1 & 0\\ 0 & 0 & -2\omega_0 & 1\\ 0 & 0 & \omega_0^2 & 0\end{bmatrix}}_{A}x + \underbrace{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}}_{B}u + \underbrace{\begin{bmatrix}0\\0\\\omega_0\\0\end{bmatrix}}_{N}n\\ y &= \underbrace{\begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}}_{C}x + w.\end{aligned}$$

d. From equation 5.76-77 in the book we see that we need to find the symmetric positive definite solution P to the Riccati equation

$$AP + PA^T - PC^TR_2^{-1}CP + \mathbf{N}R_1\mathbf{N^T} = 0,$$

where $R_1 = 1$, $R_2 = 2$, and A, B, C, N defined as above.

5.

a. The inputs to the controller are r and y_0, i.e.

$$y = \begin{pmatrix}r\\ y_0\end{pmatrix}.$$

The input to P is $\begin{pmatrix}w\\ u\end{pmatrix}$, which contains 4 signals, and the output is $\begin{pmatrix}z\\ y\end{pmatrix}$, which contains 4 signals as well. Thus P must be 4×4.

b. We know that

$$\begin{pmatrix} z \\ y \end{pmatrix} = \begin{pmatrix} e \\ u \\ r \\ y_0 \end{pmatrix}, \qquad \begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} d \\ n \\ r \\ u \end{pmatrix}.$$

The block diagram gives that

$$\begin{aligned} e &= r - x = r - P_0(d+u), \\ u &= u, \\ r &= r, \\ y_0 &= n + P_0(d+u). \end{aligned}$$

Arranging this into matrix form gives the answer:

$$P = \begin{pmatrix} -P_0 & 0 & 1 & -P_0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ P_0 & 1 & 0 & P_0 \end{pmatrix}.$$

c. The control objective a) is convex in H, and H is a linear function of Q, so the control objective a) is convex in Q. Since it is satisfied for Q_1 and Q_2, it is thus satisfied for any convex combination

$$Q = wQ_1 + (1-w)Q_2, \qquad w \in [0,1].$$

We see from the impulse responses that neither Q_1 nor Q_2 satisfies b) and c). However, a convex combination of Q_1 and Q_2 will give the same convex combination of the disturbance responses. Taking e.g. $w = 0.3$,

- The control signal satisfies $|u(t)| \le 0.3{\cdot}0.4{+}0.7{\cdot}2 = 1.52$, since $|u(t)| \le 0.4$ with C_1 and $|u(t)| \le 2$ with C_2.
- When $t \ge 3$, the control error satisfies $|e(t)| \le 0.3 \cdot 0.7 + 0.7 \cdot 0.05 = 0.275$, since $|e(t)| \le 0.7$ with C_1 and $|e(t)| \le 0.05$ with C_2.

Thus we can use $Q = 0.3Q_1 + 0.7Q_2$.

6.

a. The determinant is $\frac{1}{(s+1)(s+2)} - \frac{2}{(s+2)^2} = \frac{-s}{(s+1)(s+2)^2}$, which together with the subdeterminants means that **the poles are** $s = -2, -2, -1$**.** The determinant is the maximal subdeterminant so **the zero is** $s = 0$.

b. A state space form can be found by rewriting the system as

$$\begin{aligned} G(s) &= \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ \frac{2}{s+2} & \frac{1}{s+2} \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} \\ \frac{2}{s+2} \end{bmatrix} [1 \quad 0] + \begin{bmatrix} \frac{1}{s+2} \\ \frac{1}{s+2} \end{bmatrix} [0 \quad 1] = \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} [\, \tfrac{1}{s+1} \quad \tfrac{1}{s+2} \,] \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{s+2} [0 \quad 1] \end{aligned}$$

$$\begin{aligned}\dot{x} &= \begin{bmatrix}-1 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -2\end{bmatrix} x + \begin{bmatrix}1 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix} u\\ y &= \begin{bmatrix}1 & 0 & 1\\ 0 & 2 & 1\end{bmatrix} x\end{aligned}$$

c. For constant inputs ($s = 0$) the transfer function becomes $G(0) = \begin{bmatrix}1 & 1/2\\ 1 & 1/2\end{bmatrix}$ which means $y_1(t) = y_2(t)$ in stationarity.

7. The small gain theorem gives that the closed-loop system is stable if

$$\| - C/(1+PK)\|_\infty \|\Delta\|_\infty < 1,$$

this is equivalent to

$$\| - C/(1+PK)\|_\infty = \|K/(1+K/(s+1))\|_\infty < 1/2.$$

Where the largest gain is obtained for obtained for $\omega = \infty$, this gives the constraint

$$K < 1/2.$$

8.

a. For $\gamma = 0.1$

$$G(0) = \begin{bmatrix}10 & 0.2\\ 0.1 & 2\end{bmatrix}$$

this gives

$$\text{RGA}(0) = \begin{bmatrix}1.001 & -0.001\\ -0.001 & 1.001\end{bmatrix}.$$

For $\gamma = 1$

$$G(0) = \begin{bmatrix}1 & 2\\ 1 & 0.2\end{bmatrix}$$

this gives

$$\text{RGA}(0) = \begin{bmatrix}-0.11 & 1.11\\ 1.11 & -0.11\end{bmatrix},$$

clearly, y_1 should be paired with u_1 for $\gamma = 0.1$

b. With $\gamma = 1$,

$$G(0) = \begin{bmatrix}1 & 2\\ 1 & 0.2\end{bmatrix}.$$

With the decoupling matrices W_1 and W_2 where $W_2 = G^{-1}(0)$, $W_1 = I$, $\tilde{G}(0) = W_2 G(0) W_1 = I$ is decoupled,

$$G^{-1}(0) = -\frac{1}{1.8}\begin{bmatrix}0.2 & -2\\ -1 & 1\end{bmatrix}.$$

9.

a. The controllability gramian S and the observability gramian O are given by the solution to the Lyapunov equations

$$AS + SA^T + BB^T = 0$$
$$A^TO + OA + C^TC = 0$$

Since $A = A^T$ and $B = C^T$, this reduces to only solving one of the Lyapunov equations. If the realization is balanced, this amounts to finding a solution in the form

$$S = O = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

The terms of the Lyapunov equation then gives the following set of equations

$$\begin{aligned} -4\sigma_1 + \frac{1}{4} &= 0 \\ \sigma_1 + \sigma_2 + \frac{1}{2}(-1 - \frac{\sqrt{2}}{2}) &= 0 \\ -4\sigma_2 + \frac{1}{4} + (-1 - \frac{\sqrt{2}}{2})2 &= 0 \end{aligned}$$

with solution $\sigma_1 = 1/16 = 0.0625$ and $\sigma_2 = \frac{7+4\sqrt{2}}{16} \approx 0.7911$. Hence, the realization is balanced.

b. The smallest Hankel singular value is σ_1. This corresponds to eliminating ξ_1:

$$\begin{aligned} 0 = -2\xi_1 + \xi_2 + \frac{1}{2}y_2 \Longrightarrow \\ \xi_1 = \frac{1}{2}\xi_2 + \frac{1}{4}y_2 \end{aligned}$$

Inserting this into the rest of the system equations gives

$$\begin{aligned} \dot{\xi}_2 &= \xi_1 - 2\xi_2 + \frac{1}{2}y_1 + (-1 - \frac{\sqrt{2}}{2})y_2 \\ &= \frac{1}{2}\xi_2 + +\frac{1}{4}y_2 - 2\xi_2 + \frac{1}{2}y_1 + (-1 - \frac{\sqrt{2}}{2})y_2 \\ &= -\frac{3}{2}\xi_2 + \frac{1}{2}y_1 - \frac{3 + 2\sqrt{2}}{4}y_2 \\ u_1 &= \frac{1}{2}\xi_2 \\ u_2 &= \frac{1}{2}\xi_1 + (-1 - \frac{\sqrt{2}}{2})\xi_2 = \frac{1}{4}\xi_2 + \frac{1}{8}y_2 + (-1 - \frac{\sqrt{2}}{2})\xi_2 \\ &= -\frac{3 + 2\sqrt{2}}{4}\xi_2 + \frac{1}{8}y_2 \end{aligned}$$

or in matrix form

$$\begin{aligned} \dot{\xi}_2 &= -\frac{3}{2}\xi_2 + \begin{pmatrix} \frac{1}{2} & -\frac{3+2\sqrt{2}}{4} \end{pmatrix} y = A\xi_2 + By \\ u &= \begin{pmatrix} \frac{1}{2} \\ -\frac{3+2\sqrt{2}}{4} \end{pmatrix} \xi_2 + \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{8} \end{pmatrix} y = C\xi_2 + Dy \end{aligned}$$