LUND UNIVERSITY

Department of
**AUTOMATIC CONTROL**

# FRTN10 Multivariable Control
## Exam 2016-05-13

### Grading
All answers must include a clear motivation and a well-formulated answer. Answers may be given in English or Swedish. The total number of points is 25. The maximum number of points is specified for each subproblem. In several cases the subproblems can be solved independently.

### Accepted aid
The textbook *Glad & Ljung*, standard mathematical tables like TEFYMA, an authorized “Formelsamling i Reglerteknik”/”Collection of Formulas” and a pocket calculator. Handouts of lecture notes and lecture slides are also allowed.

### Results
The results will be reported via LADOK.

**1.** A system with one input $u$ and two outputs $y_1, y_2$ is given by

$$\begin{cases} \dot{y}_1 + y_1 + 2y_2 &= \dot{u} + 3u \\ \dot{y}_2 + 4y_1 &= u \end{cases}$$

**a.** Find the system transfer matrix. (2 p)

**b.** Find a system state-space representation. (1 p)

*Solution*

**a.** Introduce

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

This gives

$$\dot{y} + a_1 y = b_1 \dot{u} + b_2 u,$$

with

$$a_1 = \begin{pmatrix} 1 & 2 \\ 4 & 0 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Laplace transformation gives

$$(sI + a_1)Y(s) = (sb_1 + b_2)U(s).$$

We get

$$Y(s) = (sI + a_1)^{-1}(sb_1 + b_2)U(s)$$

that is,

$$G(s) = \begin{pmatrix} \frac{s^2+3s-2}{s^2+s-8} \\ \frac{-3s-11}{s^2+s-8} \end{pmatrix} = \begin{pmatrix} \frac{2s+6}{s^2+s-8} + 1 \\ \frac{-3s-11}{s^2+s-8} \end{pmatrix}.$$

**b.** A state space representation could be given on the controllable canonical form, as

$$\dot{x}(t) = \begin{pmatrix} -1 & 8 \\ 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t),$$

$$y(t) = \begin{pmatrix} 2 & 6 \\ -3 & -11 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t).$$

**2.** You want to design a controller for the process

$$P(s) = \frac{s-1}{(s-5)(s+2)}.$$

An objective of the control design is to keep down the magnitude of the sensitivity function $S(s)$. A frequency-dependent upper bound on the magnitude of the sensitivity function has been specified as

$$|S(i\omega)| \leq |F(i\omega)|, \quad \forall \omega$$

where

$$F(s) = \frac{2s}{s+2}.$$

**a.** Find a weighting function $W(s)$ such that the constraint can be formulated as

$$\sup_{\omega} |W(i\omega)S(i\omega)| \leq 1.$$

(1 p)

**b.** Is it possible to find a stabilizing controller that satisfies the constraint? (1 p)

*Solution*

**a.** The constraint can be rewritten as

$$|S(i\omega)F^{-1}(i\omega)| \leq 1 \quad \forall\omega$$

i.e.

$$\sup_{\omega} |F^{-1}(i\omega)S(i\omega)| \leq 1$$

so we have $W(s) = F^{-1}(s) = \frac{s+2}{2s}$.

**b.** A constraint of the type

$$\sup_{\omega} |W_a(i\omega)S(i\omega)| \leq 1$$

where $W_a(s) = \frac{s+a}{2s}$ is impossible to satisfy unless $|W_a(z)| \leq 1$ for all RHP zeros $s = z$ of the process, i.e. $a \leq z$ for all such zeros. This comes from the fact that

$$S(z) = \frac{1}{1 + P(z)C(z)} = 1$$

since

$$P(z) = 0.$$

Here we have $a = 2$, and there is an RHP zero at $s = 1$. Therefore, it is not possible to satisfy the constraint on $S(s)$ with any stabilizing controller.

**3.** In the first lab of this course, you designed a controller for the mass-spring process whose Bode diagram is shown in Figure 1. Figure 2 shows the Bode diagrams of the open loop $L(s) = P(s)C(s)$ for six different controllers $C_1(s)$ to $C_6(s)$. Figure 3 shows the results from using the six controllers on the process. In these simulations the process is originally at rest. At time $t = 5$ $s$ the reference value is changed to $r = 1$. At time $t = 30$ $s$, a load disturbance in the form of a step is applied to the process.

Pair the six step responses $A$ to $F$ to the six controllers 1 to 6. You must motivate your answers. (3 p)

*Solution*

**a.** Step responses A and B have static errors in the load disturbance response, which is caused by not having an integrator in the controller. If there is no integrator, the low-frequency phase of $L(s)$ should be the same as that for $P(s)$, i.e., $-90°$ as in controllers 2 and 6. Step response B is faster than A, which corresponds to a higher cross-over frequency as for controller 6.

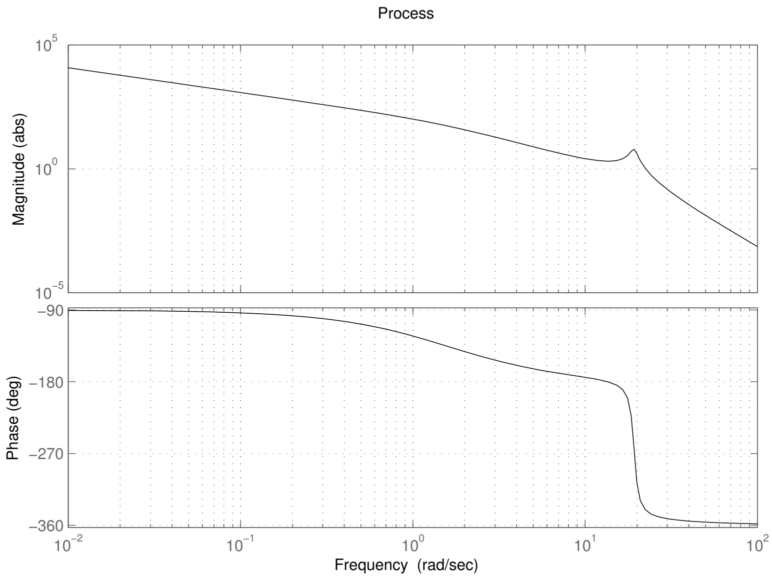



**Figure 1** The Bode diagram of the mass-spring process in Problem 3.

Thus, B → 6, A → 2.

Step response D is unstable, which corresponds to a negative phase margin. The phase of the open-loop system for controller 1 is below $-180°$ at the cross-over frequency, so the closed-loop system cannot be stable.

Thus, D → 1.

Step response F has a slow response to the load disturbance, which corresponds to a small low-frequency gain. Of the remaining controllers 3, 4, 5, the low-frequency gain is smallest for controller 5.

Thus, F → 5.

The remaining controllers 3 and 4 have cross-over frequencies at approximately $\omega_{c3} = 2.5$ and $\omega_{c4} = 5.5$. Higher cross-over frequency gives a faster response, so controller 4 corresponds to step response E and controller 3 to step response C.

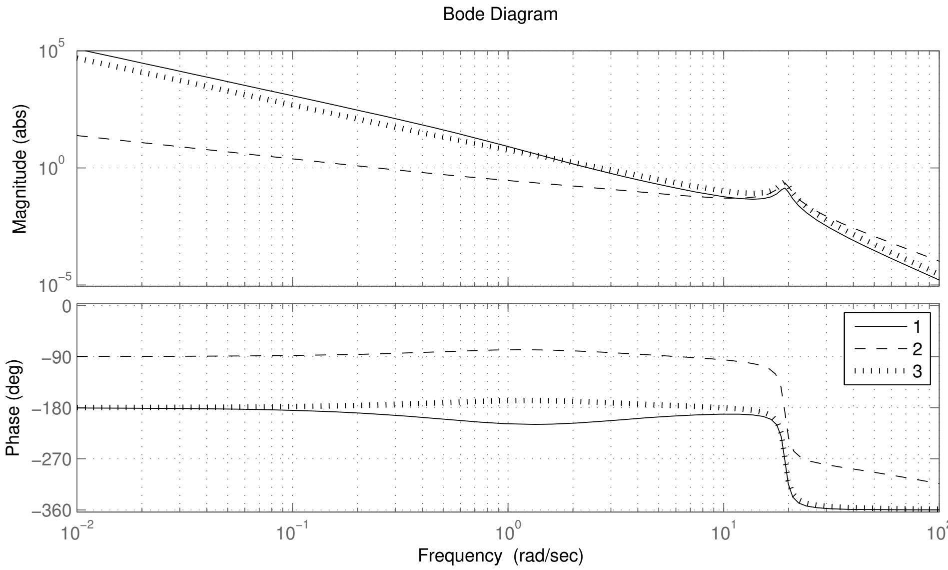
Thus, E → 4, C → 3.

**4.** In diesel engines, it is important to keep emissions of soot and oxides of Nitrogen low. Experiments have shown that emissions can be predicted from two measured signals: the combustion phasing $y_1$ and the ignition delay $y_2$. A weighted sum of emissions can be approximated by a quadratic cost function $J_y = y(t)^T Q_y y(t)$ with level curves shown in Figure 4.

**a.** Two different simulations of the engine are shown in Figure 5. Which of the two cases would correspond to lower emissions? (1 p)

**b.** We can control the system using two control signals: the start of injection angle $u_1$ and the exhaust gas recirculation valve position $u_2$. To minimize emissions we want to minimize the cost function

$$J = \int_0^\infty (y^T Q_y y + u^T Q_u u) dt$$



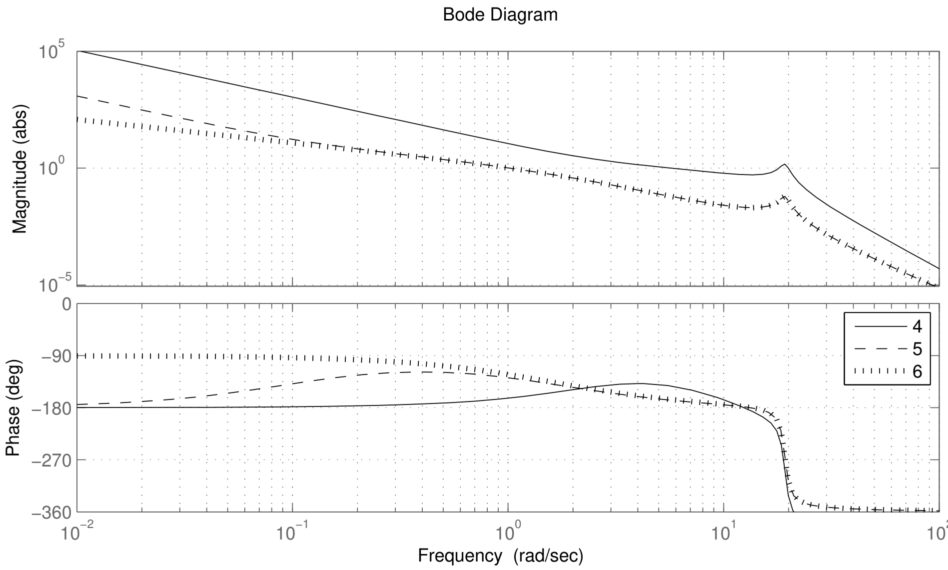




**Figure 2** The Bode diagrams of the the open-loop system $L(s) = P(s)C(s)$ using six different controllers in Problem 3.

where $Q_y = \begin{pmatrix} 6 & 20 \\ 20 & 100 \end{pmatrix}$ and $Q_u = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

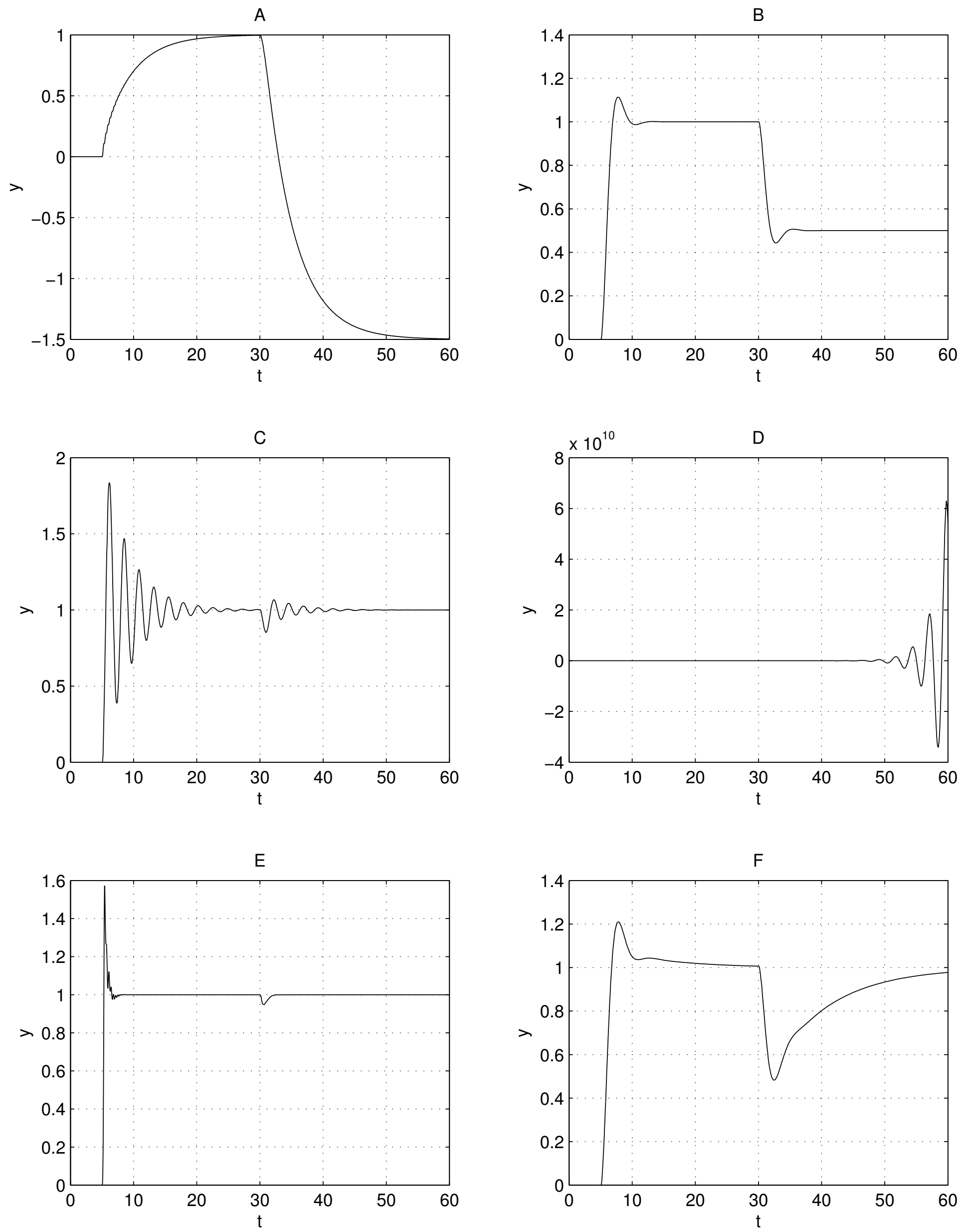



**Figure 3** The step responses of the closed-loop system using the six controllers in Problem 3. At $t = 5\ s$, the reference value is changed. At $t = 30\ s$, a load disturbance enters the system.

A continuous-time model of the engine dynamics is given by

$$\dot{x} = \begin{pmatrix} -33 & -31 \\ 31 & -33 \end{pmatrix} x + \begin{pmatrix} 2 & 0.08 \\ -2 & -0.08 \end{pmatrix} u$$

$$y = \begin{pmatrix} 16 & 1 \\ -2 & 0 \end{pmatrix} x$$

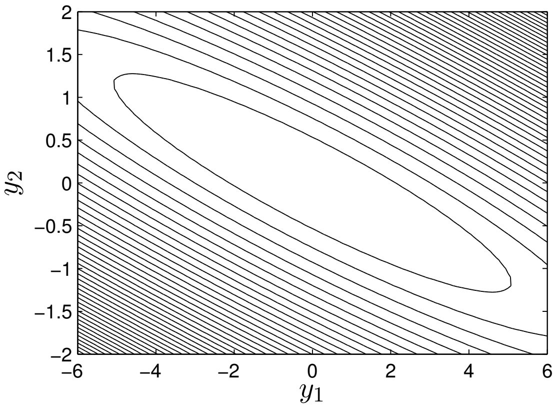



**Figure 4** Level curves of $J_y = y(t)^T Q_y y(t)$ in Problem 4.

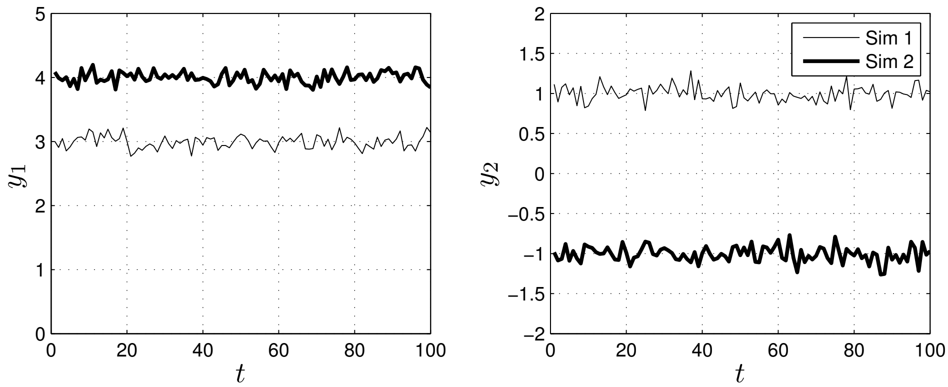



**Figure 5** Results of two simulations of the engine, Problem 4.

The optimal control law assuming full state feedback is $u = -Lx$. Give the equations that are needed to compute the optimal feedback matrix $L$. Define all variables included in the equations. You do not need to solve the equations. (2 p)

*Solution*

**a.** In the first simulation, $y_1$ and $y_2$ are close to the point $(3, 1)$, and in the second simulation to $(4, -1)$. Relating these points to the level curve plot, we see that $(4, -1)$ corresponds to a lower cost. Thus, simulation 2 results in lower emissions.

**b.** The system is given on the form

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

and the cost function can be rewritten as

$$J = \int_0^\infty y^T Q_y y + u^T Q_u u dt = \int_0^\infty x^T \underbrace{C^T Q_y C}_{Q_1} x + u^T \underbrace{Q_u}_{Q_2} u dt.$$

The optimal feedback gain $L$ is found by finding the positive definite solution $S$ to the Riccati equation

$$A^T S + SA + C^T Q_y C - SBQ_u^{-1} B^T S = 0$$

and computing $L$ as

$$L = Q_u^{-1} B^T S.$$

**5.** A system is given by

$$\dot{x} = -x + u + v_1$$

where $v_1$ is white noise with intensity $R_1 = 10$.

**a.** We use a sensor to measure the state $x$, but unfortunately the measurement noise cannot be neglected so we have the measurement equation

$$y = x + v_2$$

where $v_2$ is white noise with intensity $R_2 = 1$, uncorrelated to $v_1$. We decide to use a Kalman filter

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$$

to get an estimate $\hat{x}$ of the state $x$ from the noisy measurements $y$. Determine the optimal filter gain $K$, and the steady-state variance of the estimation error $\tilde{x} = x - \hat{x}$. (2 p)

**b.** We conclude that the estimation error variance found in (a) is too large. We come up with two options to solve the problem

1. Get a better sensor. This sensor costs 1000 SEK and gives intensity $R_2 = 0.5$ of the measurement noise.
2. Use a combination of two sensors of the original type. These sensors cost 100 SEK each. You can assume that measurement noise from the two sensors is uncorrelated.

Show how you can use a Kalman filter to estimate $x$ from measurements $y_1$ and $y_2$ from the two sensors in the second option. (2 p)

**c.** Compute the variance of the estimation error for the two options in b). Determine the best option. (2 p)

*Solution*

**a.** The optimal filter gain $K$ is given by

$$K = (PC^T + NR_{12})R_2^{-1}$$

where $P$ is the positive definite solution to the Riccati equation

$$AP + PA^T - (PC^T + NR_{12})R_2^{-1}(PC^T + NR_{12})^T + NR_1N^T = 0.$$

We have $A = -1$, $C = 1$, $N = 1$, $R_1 = 10$, $R_2 = 1$, $R_{12} = 0$, which gives the equation in $P$

$$P^2 + 2P - 10 = 0$$

with solutions

$$P = -1 \pm \sqrt{11}$$

with the unique positive solution $P = -1 + \sqrt{11} \approx 2.32$.

Thus, $K = P = 2.32$. The variance of the estimation error for the optimal Kalman filter is equal to $P$, so $var(\tilde{x}) = 2.32$.

**b.** With $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, the system can be written as

$$\begin{aligned} \dot{x} &= -x + u + v_1 \\ y &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} \end{aligned}$$

The difference from the system in (a) is that $C = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $R_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. A standard Kalman filter can be used to estimate $x$ from $y$, and the optimal gain $K$ is computed as in (a) with the modified $C$ and $R_2$ matrices.

**c.** *Option 1:*
With $R_2 = 0.5$, the Riccati equation in $P$ becomes

$$P^2/0.5 + 2P - 10 = 0,$$

which gives $P_1 = 1.79$.
*Option 2:*
With $C = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $R_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the Riccati equation in $P$ becomes

$$2P^2 + 2P - 10 = 0,$$

which is the same equation as in Option 1, thus giving $P_2 = P_1 = 1.79$.
Since the estimation error variance is equal to $P$ it is the same for the two options. Option 2 is much cheaper than option 1 and is probably the best choice.

**6.** We want to design a controller $C$ for the SISO control loop in Figure 6 using Youla parametrization and convex optimization. To do this, the control loop must first be transformed into the standard form of Figure 7, where $z$ are the signals that we want to control, $y$ are the signals available to the controller, $w$ are the exogenous inputs and $u$ is the control signal.

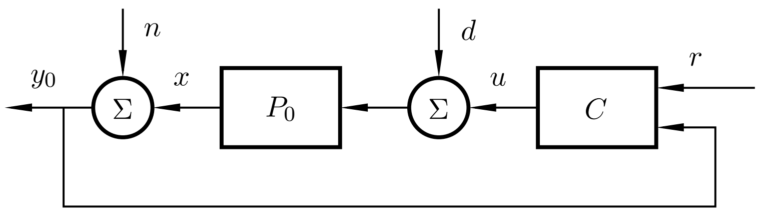



**Figure 6** The control loop in problem 6

The signals $z$ and $w$ are given by

$$z = \begin{pmatrix} e \\ u \end{pmatrix}, \qquad w = \begin{pmatrix} d \\ n \\ r \end{pmatrix},$$

where the control error is $e = r - x$. The controller $C$, which is a $1 \times 2$ transfer function, is the same in both figures, as is the control signal $u$.

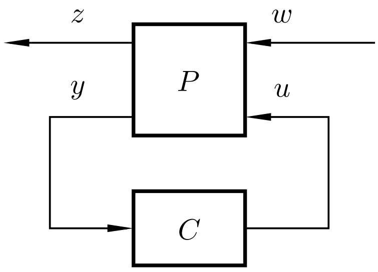



**Figure 7** Desired form of the control loop in problem 6

**a.** What is the controller input $y$ of Figure 7 according to Figure 6?
What is the size of the transfer function matrix $P$?

(1 p)

**b.** Find the transfer function matrix $P$ so that Figure 7 and Figure 6 describe the same control problem. (2 p)

**c.** The Youla parametrization results in the closed loop system

$$z = Hw$$

where the transfer function $H$ is given by

$$H = P_{zw} + P_{zu}QP_{yw}.$$

The control objective is

a) To make the $\mathcal{L}_2$-gain $||H_{ij}||_2 \leq 10$ for all elements $H_{ij}$.

b) During an impulse disturbance experiment in $d$, the control signal should satisfy $|u(t)| \leq 1$.

c) During an impulse disturbance experiment in $d$, from two seconds onward the control error should be small: $|e(t)| \leq 0.75, t \geq 2$ if the impulse occurs at $t = 0$.

Two transfer functions $Q_1$ and $Q_2$ have been found that satisfy objective a). Figure 8 shows impulse responses from $d$ to $e$ and $u$ when using the corresponding controllers $C_1$ and $C_2$. Find a $Q$ that satisfies all three objectives a),b),c). (2 p)

*Solution*

**a.** The inputs to the controller are $r$ and $y_0$, i.e.

$$y = \begin{pmatrix} r \\ y_0 \end{pmatrix}.$$

The input to $P$ is $\begin{pmatrix} w \\ u \end{pmatrix}$, which contains 4 signals, and the output is $\begin{pmatrix} z \\ y \end{pmatrix}$, which contains 4 signals as well. Thus $P$ must be $4 \times 4$.

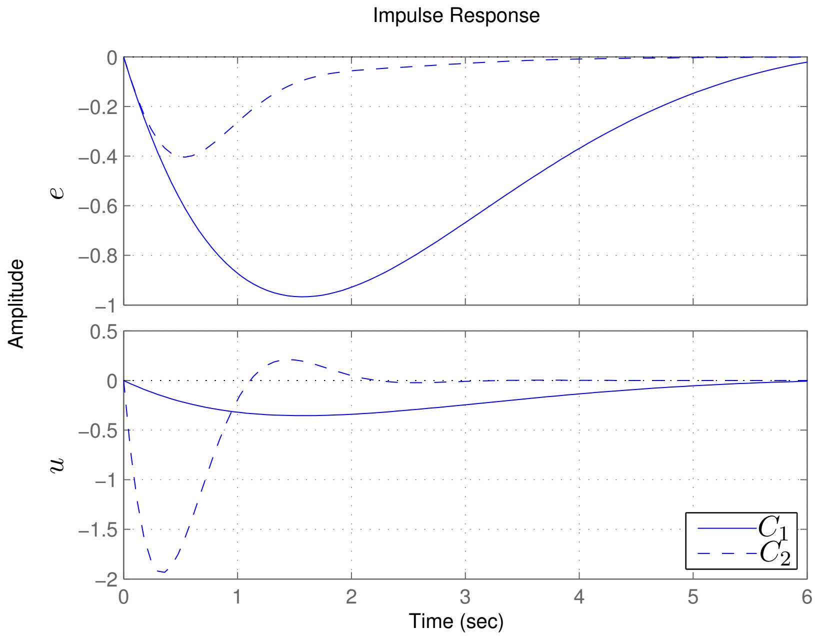



**Figure 8** Impulse responses from disturbance $d$ to control error $e$ (top) and control signal $u$ (bottom) for the controllers $C_1$ and $C_2$ in problem 6

**b.** We know that

$$\begin{pmatrix} z \\ y \end{pmatrix} = \begin{pmatrix} e \\ u \\ r \\ y_0 \end{pmatrix}, \qquad \begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} d \\ n \\ r \\ u \end{pmatrix}.$$

The block diagram gives that

$$\begin{aligned} e &= r - x = r - P_0(d + u), \\ u &= u, \\ r &= r, \\ y_0 &= n + P_0(d + u). \end{aligned}$$

Arranging this into matrix form gives the answer:

$$P = \begin{pmatrix} -P_0 & 0 & 1 & -P_0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ P_0 & 1 & 0 & P_0 \end{pmatrix}.$$

**c.** The control objective a) is convex in $H$, and $H$ is a linear function of $Q$, so the control objective a) is convex in $Q$. Since it is satisfied for $Q_1$ and $Q_2$, it is thus satisfied for any convex combination

$$Q = wQ_1 + (1 - w)Q_2, \qquad w \in [0, 1].$$

We see from the impulse responses that neither $Q_1$ nor $Q_2$ satisfies b) or c). However, a convex combination of $Q_1$ and $Q_2$ will give the same convex combination of the disturbance responses. Taking e.g. $w = 0.7$,

- The control signal satisfies $|u(t)| \leq 0.7 \cdot 0.4 + 0.3 \cdot 2 = 0.88$, since $|u(t)| \leq 0.4$ with $C_1$ and $|u(t)| \leq 2$ with $C_2$.
- When $t \geq 2$, the control error satisfies $|e(t)| \leq 0.7 \cdot 1 + 0.3 \cdot 0.1 = 0.73$, since $|e(t)| \leq 1$ with $C_1$ and $|e(t)| \leq 0.1$ with $C_2$.

Thus we can use $Q = 0.7Q_1 + 0.3Q_2$.

**7.** Consider the control system in the block diagram in Figure 9. The process $P(s)$ is modeled to have an unstructured uncertainty $\Delta$.

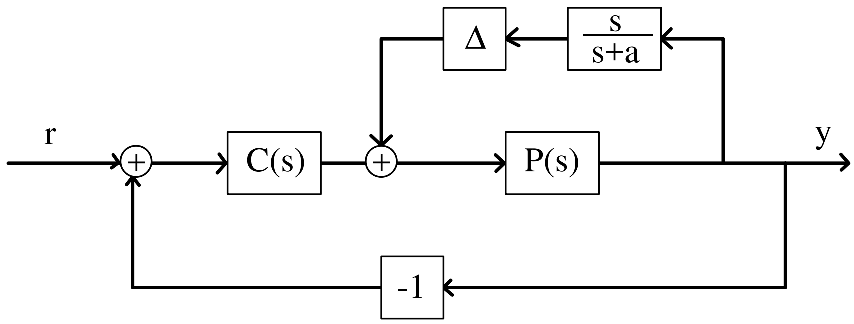



**Figure 9** Block diagram for the system in problem 7.

**a.** Explain the role of the transfer function $\frac{s}{s+a}$ $(a > 0)$ in the block diagram. (1 p)

**b.** Give an expression for $\gamma^*$ such that stability of the closed loop system is guaranteed by the small gain theorem if and only if $\|\Delta\|_\infty < \gamma^*$. Assume that the nominal system (i.e., with $\Delta = 0$) is asymptotically stable. (2 p)

*Solution*

**a.** It is a weighting function giving the unstructured uncertainty some structure. In this case it says that the uncertainty is small for low frequencies, less than $a$, and large for high frequencies.

**b.** By calling the input to the uncertainty block $v_1$ and the output from it $v_2$, the system can be rewritten according to Figure 10. To derive $G(s)$, first calculate $Y(s)$:

$$Y(s) = P(s)(V_2 - C(s)Y(s))$$
$$\Rightarrow Y(s) = \frac{P(s)}{1 + C(s)P(s)} V_2(s)$$

Now $V_1(s)$ is given by

$$V_1(s) = \frac{s}{s+a} Y(s) = \frac{P(s)\frac{s}{s+a}}{1 + C(s)P(s)} V_2(s)$$

and the transfer function $G(s)$ is hence given by

$$G(s) = \frac{P(s)\frac{s}{s+a}}{1 + C(s)P(s)}$$

The closed loop system is stable according to the small gain theorem if and only if $\|\Delta\| < \gamma^*$ where $\gamma^* = \|G(i\omega)\|_\infty^{-1}$.

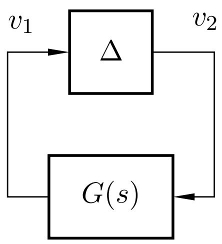



**Figure 10** Equivalent block diagram for the system in problem 7.