LUND INSTITUTE
OF TECHNOLOGY
Lund University

Department of
AUTOMATIC CONTROL

Multivariable Control (FRTN10)

Exam October 24, 2012, hours: 8.00-13.00

Points and grades

All answers must include a clear motivation and a well-formulated answer. Answers may be given in English or Swedish. The total number of points is 25. The maximum number of points is specified for each subproblem.

Accepted aid

The textbook *Glad & Ljung*, standard mathematical tables like TEFYMA, an authorized "Formelsamling i Reglerteknik"/"Collection of Formulas" and a pocket calculator. Handouts of lecture notes and lecture slides are also allowed.

Results

The result of the exam will be posted on the notice-board at the Department. The result as well as solutions will be available on the course home page:
`http://www.control.lth.se/Education/EngineeringProgram/FRTN10.html`

1. Consider a system with inputs (u_1, u_2) and outputs (y_1, y_2), given by the differential equations

$$\ddot{y}_1 + 2\dot{y}_1 - y_2 + y_1 = u_1 + \dot{u}_2 + 2u_2$$
$$y_2 + \dot{y}_1 + y_1 = u_1 - 2u_2$$

Calculate the transfer matrix of the system. (2 p)

Solution

Start by applying the Laplace transformation to both sides of the equations:

$$s^2Y_1 + 2sY_1 - Y_2 + Y_1 = U_1 + sU_2 + 2U_2$$
$$Y_2 + sY_1 + Y_1 = U_1 - 2U_2$$

or equivalently in matrix form:

$$\begin{pmatrix} (s+1)^2 & -1 \\ s+1 & 1 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 1 & s+2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$$

The transfer function matrix is then given by

$$\begin{aligned} G(s) &= \begin{pmatrix} (s+1)^2 & -1 \\ s+1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & s+2 \\ 1 & -2 \end{pmatrix} \\ &= \frac{1}{(s+1)(s+2)} \begin{pmatrix} 1 & 1 \\ -(s+1) & (s+1)^2 \end{pmatrix} \begin{pmatrix} 1 & s+2 \\ 1 & -2 \end{pmatrix} \\ &= \frac{1}{(s+1)(s+2)} \begin{pmatrix} 2 & s \\ s(s+1) & -3(s+1)(s+\frac{4}{3}) \end{pmatrix} \\ &= \begin{pmatrix} \frac{2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \\ \frac{s}{s+2} & -\frac{3s+4}{s+2} \end{pmatrix} \end{aligned}$$

2. Consider the system

$$G(s) = \begin{pmatrix} \frac{\alpha}{s+1} & -\frac{s+2}{s+1} \\ \frac{1}{s+1} & \frac{1}{s+2} \end{pmatrix}$$

a. What are the poles of the system? For which values of α is the system non-minimum phase? (2 p)

b. Derive a state space realization of the system. (2 p)

Solution

a. The determinant of $G(s)$ is given by

$$\det G(s) = \frac{\alpha}{(s+1)(s+2)} + \frac{s+2}{(s+1)^2} = \frac{s^2 + (4+\alpha)s + 4 + \alpha}{(s+1)^2(s+2)}$$

Thus, the poles are located in $s = -1$ (multiplicity 2) and $s = -2$ (multiplicity 1). The transmission zeros are located at the roots of the zero polynomial

$$s^2 + (4+\alpha)s + 4 + \alpha$$

The system is non-minimum phase when $4 + a < 0$, i.e. when $a < -4$.

b. A diagonal state space realization can be derived by noting that

$$\begin{aligned}
\begin{pmatrix} \frac{\alpha}{s+1} & -\frac{s+2}{s+1} \\ \frac{1}{s+1} & \frac{1}{s+2} \end{pmatrix} &= \begin{pmatrix} \frac{\alpha}{s+1} & -1-\frac{1}{s+1} \\ \frac{1}{s+1} & \frac{1}{s+2} \end{pmatrix} \\
&= \frac{1}{s+1}\begin{pmatrix} \alpha & -1 \\ 1 & 0 \end{pmatrix} + \frac{1}{s+2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \\
&= \frac{1}{s+1}\begin{pmatrix} \alpha & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{s+2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} + D \\
&= \frac{1}{s+1}C_1B_1 + \frac{1}{s+2}C_2B_2 + D
\end{aligned}$$

Note that the pole in $s = -1$ with multiplicity 2 requires us to have a B_1 with two columns. Also note that the factorizations C_iB_i are not at all unique. Taken together, this allows us to write the corresponding diagonal state space realization as

$$\begin{aligned}
\dot{x} &= \begin{pmatrix} -p_1 I_{2x2} & 0 \\ 0 & -p_2 \end{pmatrix} x + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u \\
y &= \begin{pmatrix} C_1 & C_2 \end{pmatrix} x + D
\end{aligned}$$

where $p_1 = 1$ and $p_2 = 2$, or explicitly

$$\begin{aligned}
\dot{x} &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} u \\
y &= \begin{pmatrix} \alpha & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

3. Consider the control system in the block diagram in Figure 1. The process $P(s)$ is modeled to have an unstructured uncertainty Δ.

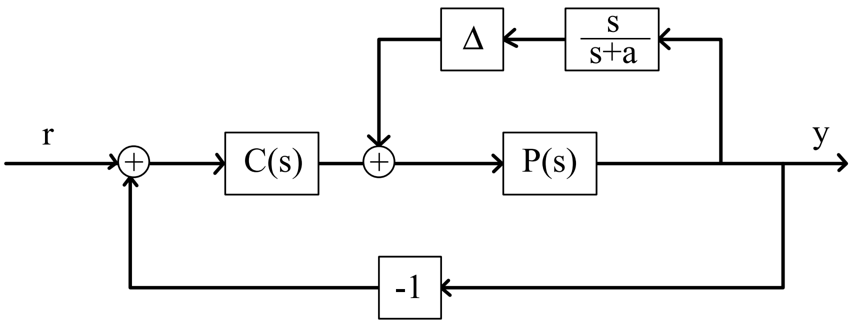

Figure 1 Block diagram for the system in problem 3.

a. Explain the role of the transfer function $\frac{s}{s+a}$ $(a > 0)$ in the block diagram. (1 p)

b. Give an expression for γ^* such that stability of the closed loop system is guaranteed by the small gain theorem if and only if $\|\Delta\|_\infty < \gamma^*$. Assume that the nominal system (i.e. with $\Delta = 0$) is asymptotically stable. (2 p)

Solution

a. It is a weighting function giving the unstructured uncertainty some structure. In this case it says that the uncertainty is small for low frequencies, less than a, and large for high frequencies.

b. By calling the input to the uncertainty block v_1 and the output from it v_2, the system can be rewritten according to Figure 2. To derive $G(s)$, first calculate $Y(s)$:

$$Y(s) = P(s)(V_1 - C(s)Y(s))$$

$$\Rightarrow Y(s) = \frac{P(s)}{1 + C(s)P(s)} V_1(s)$$

Now $V_2(s)$ is given by

$$V_2(s) = \frac{s}{s+a} Y(s) = \frac{P(s)\frac{s}{s+a}}{1 + C(s)P(s)} V_1(s)$$

and the transfer function $G(s)$ is hence given by

$$G(s) = \frac{P(s)\frac{s}{s+a}}{1 + C(s)P(s)}$$

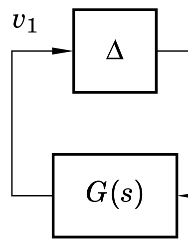

Figure 2 Equivalent block diagram for the system in problem 3.

The closed loop system is stable according to the small gain theorem if and only if $\|\Delta\| < \gamma^*$ where $\gamma^* = \|G(s)\|_\infty^{-1}$.

4. We will consider optimal state estimation for the setup shown in Figure 3 with $P(s) = \frac{6}{s+2.5}$.

a. The process disturbance v_1 has the power spectrum $\Phi_{v_1}(\omega) = \frac{1}{\omega^2+1}$. The measurement error v_2 can be considered as white noise with intensity 1; $\Phi_{v_2}(\omega) = 1$. Rewrite the system to a form where the only inputs are the control signal and white noise signals with intensity 1. (2 p)

b. Determine the optimal Kalman filter gain for the system derived in the previous subproblem.

Hint: For some choice of state variables, the Riccati equation has the solution $P = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 3/8 \end{pmatrix}$. (2 p)

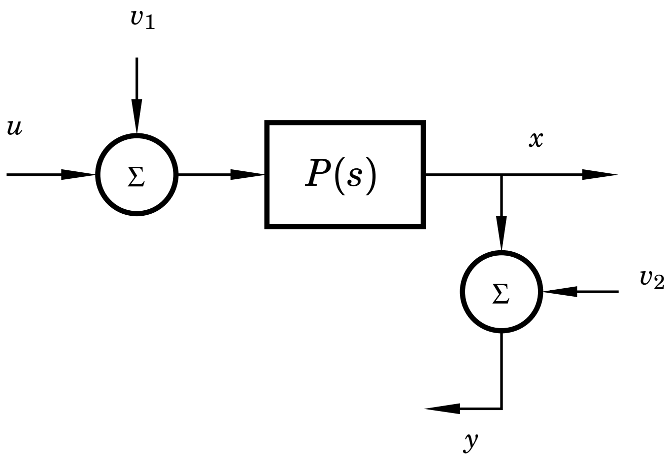

Figure 3 The setup for the system in problem 4.

Solution

a. We will here refer to the state in the system P as x_1. The output can be written as $y = x_1 + v_2$ where

$$X_1(s) = \frac{6}{s+2.5}(U(s) + V_1(s)) \Rightarrow \dot{x_1} = -2.5x_1 + 6u + 6v_1$$

$$\Phi_{v_1}(\omega) = \frac{1}{\omega^2+1} \Rightarrow v_1 = \frac{1}{s+1}e_1 \Rightarrow \dot{v_1} = -v_1 + e_1 \qquad (\Phi_{e_1}(\omega) = 1)$$

In matrix form, with $x = \begin{pmatrix} x_1 \\ v_1 \end{pmatrix}$, this becomes:

$$\dot{x} = \begin{pmatrix} -2.5 & 6 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 6 \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e_1$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x + v_2$$

b. The optimal Kalman filter gain is $K = (PC^T + NR_{12})R_2^{-1}$. First, the choice of state variables (and subsequent state space representation, including C) which corresponds to the P-matrix in the hint must be found. The state representation found in the first subproblem is inserted into the Riccati equation

$$0 = R_1 + AP + PA^T - (PC^T + R_{12})R_2^{-1}(PC^T + R_{12})^T$$
$$\Rightarrow 0 = R_1 + AP + PA^T - PC^TCP^T$$

where $R_1 = \begin{pmatrix} 0 & 1 \end{pmatrix}^T \times 1 \times \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, R_2 = 1, R_{12} = 0$

If the first subproblem has been solved as above, this becomes

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -2.5 & 6 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{8} \end{pmatrix} + \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{8} \end{pmatrix} \begin{pmatrix} -2.5 & 0 \\ 6 & -1 \end{pmatrix}$$
$$- \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{8} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{8} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix} - \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix} = 0$$

and we have therefore showed that the given P-matrix relates to the state representation $X = \begin{pmatrix} x_1 \\ v_1 \end{pmatrix}$, for which we have known A, B and C matrices. If the state representation is inverted, i.e. $X = \begin{pmatrix} v_1 \\ x_1 \end{pmatrix}$, the Riccati equation will not hold and it can be realized that the only other state representation is the aforementioned one, for which we can show as above that the Riccati equation holds.

Using the C matrix given by this state representation, we can then calculate K using the (simplified) expression

$$K = PC^T = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}$$

5. Consider control of the process with Bode diagram shown in Figure 4. For each of the following statements, determine if it is true or false. Short motivations are required! (2 p)

1. Without disturbances, a P-controller will be sufficient to follow reference steps without stationary error.
2. A step-shaped input load disturbance can be removed by a P-controller.
3. It is impossible to design a PI-controller that achieves cut-off frequency $\omega_c = 5$ rad/s and phase-margin $\varphi_m = 35$ degrees.
4. A P-controller will be sufficient to follow a ramp-shaped reference signal without stationary error.

Solution

1. True. The Bode diagram clearly shows that there is an integrator in the process, meaning that for low frequencies the sensitivity function $S(s) = 1/(1 + P(s)C(s))$ will be close to zero. In particular, $S(0) = 0$.
2. False. Because of the integrator in the process, the transfer function from input load disturbance to output at stationarity $P(0)S(0)$ will not be zero.
3. True. According to the Bode diagram of the process, we need to add phase at the desired cut-off frequency to get the specified phase margin. A PI-controller can never give a net increase in phase, so this will not be possible.

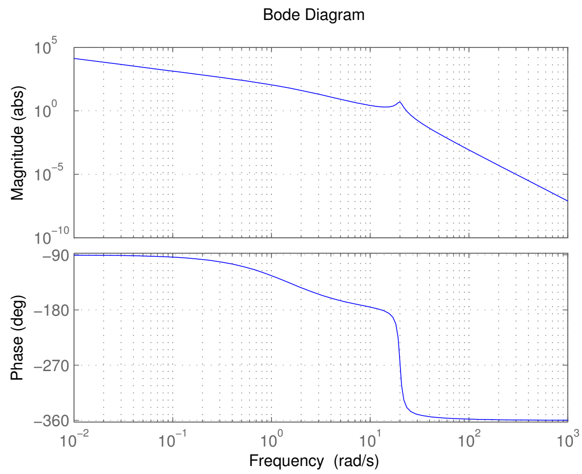

Figure 4 Bode plot for problem 5.

4. False. Assuming that the closed loop system is stable, the final value theorem says that for the error

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{1+P(s)C(s)}\frac{1}{s^2}$$
$$= \lim_{s\to 0}\frac{1}{s+sP(s)C(s)} = C \neq 0$$

where C is a nonzero constant, as the integrator in $P(s)$ will be canceled out by the factor $1/s$. Thus, there will be a stationary error.

6. Consider the following system

$$\dot{x} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ -2 & -3 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 \end{pmatrix} x + \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} x$$

Is it controllable? Is it observable? Determine the controllable and observable subspaces. Motivate! (3 p)

Solution

This problem can either be solved by studying the A, B and C matrices directly or through calculation of the observability and controllability matrices.

Using the first approach:
The dynamics for all states in the system differ from each other (there are

no multiple poles), therefore a state is controllable if the control signals can influence its value (directly or indirectly) and observable if its value can influence the measurement signals (directly or indirectly). This means that states 1, 2, 3 and 4 are controllable and states 1, 2, 4 and 5 are observable. The system as a whole is therefore neither controllable nor observable and the controllable subspace consists of all states except the 5th and the observable subspace consists of all states except the 3rd.

Using the second approach:

$$S = \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix} \qquad (n = 5)$$
$$\Rightarrow$$
$$S = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 & 8 & 0 & -26 & 0 & 80 \\ 1 & 0 & -4 & 0 & 16 & 0 & -64 & 0 & 256 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The controllability matrix has rank lower than 5 so the system is not controllable. We can see that the 5th state is not part of the controllable subspace, as the 5th row in the controllability matrix only contains zeros.

$$O = \begin{pmatrix} C \\ CA \\ AB \\ \vdots \\ CA^{n-1} \end{pmatrix} \qquad (n = 5)$$
$$\Rightarrow$$
$$O = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -5 \\ -2 & -3 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 25 \\ 8 & 9 & 0 & 0 & 25 \\ 0 & 0 & 0 & 0 & -125 \\ -26 & -27 & 0 & 0 & -125 \\ 0 & 0 & 0 & 0 & 625 \\ 80 & 81 & 0 & 0 & 625 \end{pmatrix}$$

The observability matrix has rank lower than 5 so the system is not observable. We can see that the 3rd state is not part of the observable subspace, as the 3rd column in the observability matrix only contains zeros.

7. In this problem we will study model reduction of the MIMO controller

$$\dot{\xi} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \xi + \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & -1 - \frac{\sqrt{2}}{2} \end{pmatrix} y$$

$$u = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & -1 - \frac{\sqrt{2}}{2} \end{pmatrix} \xi$$

a. Verify that the state space realization is balanced. Calculate the Hankel singular values. (2 p)

b. Based on your results from **a** perform a model reduction by eliminating the state corresponding to the smallest Hankel singular value. (2 p)

c. What is the transfer matrix of the reduced controller? (1 p)

Solution

a. The controllability gramian S and the observability gramian O are given by the solution to the Lyapunov equations

$$AS + SA^T + BB^T = 0$$
$$A^T O + OA + C^T C = 0$$

Since $A = A^T$ and $B = C^T$, this reduces to only solving one of the Lyapunov equations. If the realization is balanced, this amounts to finding a solution on the form

$$S = O = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

The terms of the Lyapunov equation then gives the following set of equations

$$-4\sigma_1 + \frac{1}{4} = 0$$
$$\sigma_1 + \sigma_2 + \frac{1}{2}(-1 - \frac{\sqrt{2}}{2}) = 0$$
$$-4\sigma_2 + \frac{1}{4} + (-1 - \frac{\sqrt{2}}{2})2 = 0$$

with solution $\sigma_1 = 1/16 = 0.0625$ and $\sigma_2 = \frac{7+4\sqrt{2}}{16} \approx 0.7911$. Hence, the realization is balanced.

b. The smallest Hankel singular value is σ_1. This corresponds to eliminating ξ_1:

$$0 = -2\xi_1 + \xi_2 + \frac{1}{2} y_2 \implies$$
$$\xi_1 = \frac{1}{2}\xi_2 + \frac{1}{4} y_2$$

Inserting this into the rest of the system equations gives

$$\begin{aligned}
\dot{\xi}_2 &= \xi_1 - 2\xi_2 + \frac{1}{2}y_1 + (-1 - \frac{\sqrt{2}}{2})y_2 \\
&= \frac{1}{2}\xi_2 + +\frac{1}{4}y_2 - 2\xi_2 + \frac{1}{2}y_1 + (-1 - \frac{\sqrt{2}}{2})y_2 \\
&= -\frac{3}{2}\xi_2 + \frac{1}{2}y_1 - \frac{3+2\sqrt{2}}{4}y_2 \\
u_1 &= \frac{1}{2}\xi_2 \\
u_2 &= \frac{1}{2}\xi_1 + (-1 - \frac{\sqrt{2}}{2})\xi_2 = \frac{1}{4}\xi_2 + \frac{1}{8}y_2 + (-1 - \frac{\sqrt{2}}{2})\xi_2 \\
&= -\frac{3+2\sqrt{2}}{4}\xi_2 + \frac{1}{8}y_2
\end{aligned}$$

or on matrix form

$$\begin{aligned}
\dot{\xi}_2 &= -\frac{3}{2}\xi_2 + \begin{pmatrix} \frac{1}{2} & -\frac{3+2\sqrt{2}}{4} \end{pmatrix} y = A\xi_2 + By \\
u &= \begin{pmatrix} \frac{1}{2} \\ -\frac{3+2\sqrt{2}}{4} \end{pmatrix} \xi_2 + \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{8} \end{pmatrix} y = C\xi_2 + Dy
\end{aligned}$$

c. Using the Laplace transform and some calculations, the transfer matrix is given by

$$G(s) = C(sI - A)^{-1}B + D = \frac{1}{s+1.5}\begin{pmatrix} \frac{1}{4} & -\frac{3+2\sqrt{2}}{8} \\ -\frac{3+2\sqrt{2}}{8} & \frac{s+10+6\sqrt{2}}{8} \end{pmatrix}$$

8. You are called in as a consultant at the company SuperControl to analyze the latest developments of one of their main rivals, UltraControl. For various reasons both companies are working hard to find good controllers for the following systems

A $P(s) = \frac{s-3}{(s+4)(s-2)}$

B $P(s) = \frac{1}{s-10}$

C $P(s) = \frac{s-1}{(s+1)^2}$

D $P(s) = \frac{1}{s^2+0.4s+4}$

UltraControl claims that they have designed stabilizing controllers that result in the sensitivity functions (S) and complementary sensitivity functions (T) in Figure 5.

It is known within SuperControl that UltraControl sometimes makes serious mistakes (their entire control group has been recruited from KTH), and your task is therefore to analyze if their control designs are reasonable. For each of the systems A–D, analyze if the given plots of S and T is even possible to achieve. Motivate your answers. (2 p)

Solution

A: The unstable pole ($p = 2$) and the unstable zero ($z = 3$) means that

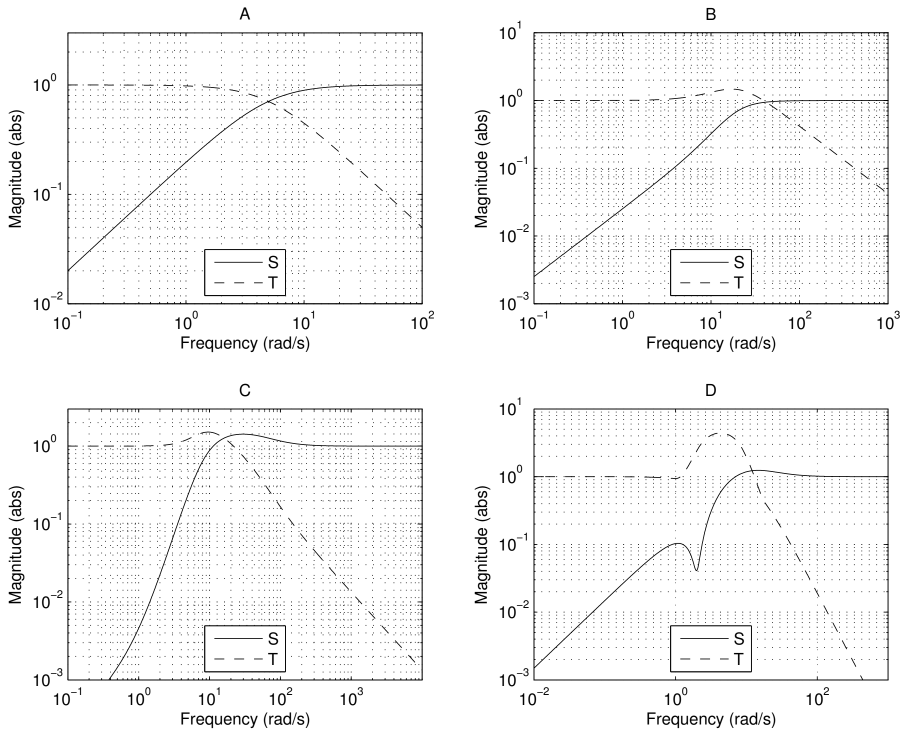

Figure 5 Plots of S and T for the different systems in problem 8.

$\|S\|_\infty \geq \left|\frac{z+p}{z-p}\right| = 5$), and as the given plot of S is less than or equal to 1 it is clear that the control design is not possible.

B: The unstable pole means that the closed loop system needs a bandwidth that is at least as fast as the pole. The system in the plot has a bandwidth of approximately 50 rad/s, and the design is hence possible.

C: The unstable zero makes it impossible to achieve a closed loop bandwidth larger than 1 rad/s, which means that the design in the plot that has a bandwidth of above 20 rad/s is not possible.

D: By using the reverse triangle inequality: $|S - T| \geq |(|S| - |T|)|$, the following inequality can be derived:

$$1 = |1| = |S + T| = |S - (-T)| \geq |(|S| - |T|)|$$

$$\Rightarrow |(|S| - |T|)| \leq 1$$

At the frequency $\omega \approx 3$ rad/s it can be seen in the plot that $|S| = 4$ and $|T| = 0.7$, but inserting this into the above relation gives $|(|S| - |T|)| = |4 - 0.7| = 3.3$, and this is clearly not less than 1, and this design is therefore also not possible.