LUND INSTITUTE OF TECHNOLOGY
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Department of
AUTOMATIC CONTROL

FRTN10 Multivariable Control

Exam 2014-04-22

Points and grades

All answers must include a clear motivation and a well-formulated answer. Answers may be given in English or Swedish. The total number of points is 25. The maximum number of points is specified for each subproblem.

Accepted aid

The textbook *Glad & Ljung*, standard mathematical tables like TEFYMA, an authorized “Formelsamling i Reglerteknik”/”Collection of Formulas” and a pocket calculator. Handouts of lecture notes and lecture slides are also allowed.

Results

The results will be reported via LADOK.

1. Consider the system

$$G(s) = [\, \tfrac{s}{s+1} \quad \tfrac{2}{s+3} \,]$$

a. Find a state space realization of the system. (2 p)

b. What are the poles and zeros of G? (1 p)

Solution

a.

$$G(s) = [\, 1 - \tfrac{1}{s+1} \quad \tfrac{2}{s+3} \,] = \frac{1}{s+1}[\, -1 \quad 0 \,] + \frac{1}{s+3}[\, 0 \quad 2 \,] + [\, 1 \quad 0 \,]$$

this gives a state space realization

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} x + \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} u$$
$$y = [\, 1 \quad 1 \,]\, x + [\, 1 \quad 0 \,]\, u.$$

b. The system can be rewritten to get a common denominator.

$$G(s) = \begin{bmatrix} \tfrac{s(s+3)}{(s+1)(s+3)} & \tfrac{2(s+1)}{(s+1)(s+3)} \end{bmatrix}$$

The pole polynomial is then

$$p(s) = (s+1)(s+3)$$

so the poles are $s = -1$ and $s = -3$, which is also easily seen from the state space realization.

The zeros are given as the greatest common divisor of the maximal minors of $G(s)$. Since the greatest common divisor of $s(s+3)$ and $2(s+1)$ is 1, the system has no zeros.

2. Consider the system

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u,$$

The system should be controlled so that J is minimized:

$$J = \int_0^\infty \left(x_1^2 + 4x_2^2 + 2x_1 u + 2u^2 \right) dt$$

a. Which of the following S solves the Riccati equation corresponding to this set-up? (Note that there can be small round-off errors.)

1. $S_1 = \begin{bmatrix} 0.3750 & 0.0115 \\ 0.2125 & 0.6215 \end{bmatrix}$
2. $S_2 = \begin{bmatrix} 0.1225 & 0.1325 \\ 0.1325 & 0.7050 \end{bmatrix}$

3. $S_3 = \begin{bmatrix} 0.1375 & 0.2025 \\ 0.2025 & 0.6550 \end{bmatrix}$

(2 p)

b. Calculate the optimal state feedback vector L so that $u = -Lx$. (1 p)

c. Redefining the controller so that it contains a reference following term, i.e. $u = -Lx + L_r r$, calculate L_r so that the static gain from r to x_1 is 1. (1 p)

Solution

a. The first alternative can be discarded as it is not symmetric. The minimization criterium can be given as $Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$, $Q_{12} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $Q_2 = 2$ in the Riccati equation $0 = A^T S + SA + Q_1 - (SB + Q_{12})Q_2^{-1}(SB + Q_{12})^T$. Calculations yield that the second alternative (S_2) is the correct answer.

b. $L = Q_2^{-1}(B^T S + Q_{12}^T) = [\,0.6225 \quad 0.1325\,]$

c.

$$L_r = \left(M\,(BL - A)^{-1}\,B\right)^{-1} =$$

$$= \left([1 \quad 0]\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}[\,0.6225 \quad 0.1325\,] - \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}\right)^{-1}\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right)^{-1} =$$

$$= 1.5.$$

3. Consider the system $Y(s) = G(s)U(s)$, where

$$G(s) = \begin{bmatrix} \frac{1}{2s+1}e^{-2s} & \frac{1}{s+1} \\ \frac{1}{4s+1} & \frac{3}{3s+1} \end{bmatrix}$$

a. Decouple $G(s)$ in stationarity using suitable decoupling matrices W_1 and W_2. Also, state how the control signal u will depend on y if the diagonalized system is controlled by $F_{diag} = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix}$. (2 p)

b. Use RGA to determine which output should be coupled to which input if two SISO controllers are to be used for controlling the system when following input variations with a frequency of 10 Hz. (2 p)

Solution

a. We want $\tilde{G}(s) = W_2(s)G(s)W_1(s)$ to be diagonal in stationarity. We can use e.g. $W_1 = W_2 = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}$ which gives

$$\tilde{G}(0) = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

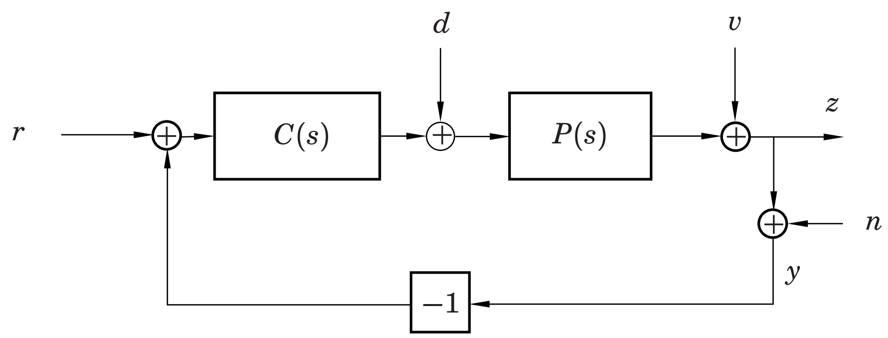

Figure 1 System in problem 4

The control signal is then given from

$$u = -W_1 F_{diag} W_2 y =$$
$$= -\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} y = \begin{pmatrix} -F_1 - F_2 & F_1 \\ F_1 & -F_1 \end{pmatrix} y.$$

b. RGA(G) = $G. * (G^{-1})^T$. Here we want to follow inputs with frequency $f =$ 10 Hz. So we want to calculate RGA at the frequency $\omega = 2\pi \cdot 10$.

$$G(20\pi i) \approx \begin{pmatrix} 0.0001 - 0.0080i & 0.0003 - 0.0159i \\ 0.0000 - 0.0040i & 0.0001 - 0.0159i \end{pmatrix}$$

which gives

$$\text{RGA}(G(20\pi i)) \approx \begin{pmatrix} 1.9997 + 0.0133i & -0.9997 - 0.0133i \\ -0.9997 - 0.0133i & 1.9997 + 0.0133i \end{pmatrix}.$$

From this we conclude that we want to control y_1 with u_1 and y_2 with u_2.

4. Consider the setup in Figure 1 with

$$P(s) = \frac{s-2}{s^2 + 8s + 16}$$

a. Show whether or not the following specification on the sensitivity function of the system can be fulfilled:

$$|S(i\omega)| \le \frac{2\omega}{\sqrt{\omega^2 + 16}} \qquad \omega \in \mathbb{R}$$

(2 p)

b. For a controller

$$C(s) = \frac{s^2 + 8s + 16}{(s+3)^2}$$

calculate the sensitivity and complementary sensitivity functions of the system. (1 p)

Solution

a. The specification

$$|S(i\omega)| \le \frac{2\omega}{\sqrt{\omega^2+16}} \qquad \omega \in \mathbb{R}^+$$

is equal to

$$\sup_{\omega} \left| \frac{\sqrt{\omega^2+16}}{2\omega} S(i\omega) \right| \le 1$$

Since

$$W_S(i\omega) = \frac{i\omega+4}{2i\omega}$$

gives

$$|W_S(i\omega)| = \frac{\sqrt{\omega^2+16}}{2\omega}$$

the specification can be written

$$\sup_{\omega} |W_S(i\omega)S(i\omega)| \le 1$$

According to Theorem 7.4 in [Glad&Ljung], this specification is impossible to meet when the process has a RHP zero in $s = z$, unless $|W_S(z)| \le 1$. Here we have a zero in $z = 2$, so

$$|W_S(z)| = \frac{2+4}{4} \ge 1$$

This means that the specification cannot be fulfilled.

b.

$$S(s) = \frac{1}{1+PC} = \frac{s^2+6s+9}{s^2+7s+7}$$
$$T(s) = \frac{PC}{1+PC} = \frac{s-2}{s^2+7s+7}$$

5. We will consider optimal state estimation for the setup shown in Figure 2 with $P(s) = \frac{6}{s+2.5}$.

a. The process disturbance v_1 has the power spectrum $\Phi_{v_1}(\omega) = \frac{1}{\omega^2+1}$. The measurement error v_2 can be considered as white noise with intensity 1; $\Phi_{v_2}(\omega) = 1$. Rewrite the system to a form where the only inputs are the control signal and white noise signals with intensity 1. (2 p)

b. Determine the optimal Kalman filter gain for the system derived in the previous subproblem.

Hint: For some choice of state variables, the Riccati equation has the solution $P = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 3/8 \end{pmatrix}$. (2 p)

Solution

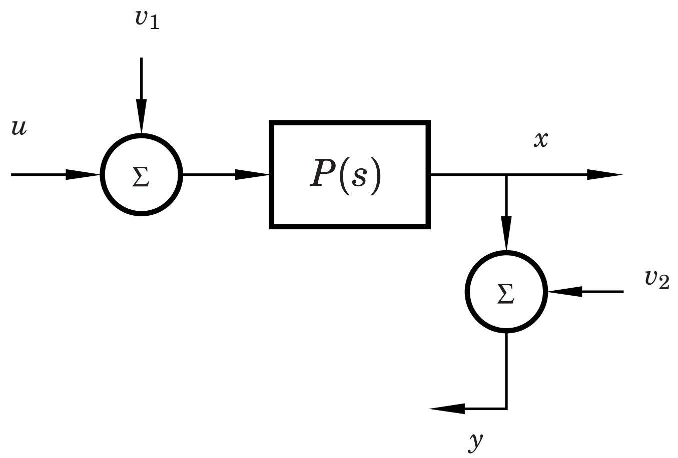

Figure 2 The setup for the system in problem 5.

a. We will here refer to the state in the system P as x_1. The output can be written as $y = x_1 + v_2$ where

$$X_1(s) = \frac{6}{s+2.5}(U(s) + V_1(s)) \Rightarrow \dot{x_1} = -2.5x_1 + 6u + 6v_1$$

$$\Phi_{v_1}(\omega) = \frac{1}{\omega^2+1} \Rightarrow v_1 = \frac{1}{s+1}e_1 \Rightarrow \dot{v_1} = -v_1 + e_1 \qquad (\Phi_{e_1}(\omega) = 1)$$

In matrix form, with $x = \begin{pmatrix} x_1 \\ v_1 \end{pmatrix}$, this becomes:

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -2.5 & 6 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 6 \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e_1 \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x + v_2 \end{aligned}$$

b. The optimal Kalman filter gain is $K = (PC^T + NR_{12})R_2^{-1}$. First, the choice of state variables (and subsequent state space representation, including C) which corresponds to the P-matrix in the hint must be found. The state representation found in the first subproblem is inserted into the Riccati equation

$$\begin{aligned} 0 &= NR_1N^T + AP + PA^T - (PC^T + R_{12})R_2^{-1}(PC^T + R_{12})^T \\ \Rightarrow 0 &= NR_1N^T + AP + PA^T - PC^TCP^T \end{aligned}$$

where $N = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$, $R_1 = 1$, $R_2 = 1$, $R_{12} = 0$.

If the first subproblem has been solved as above, this becomes

$$\begin{aligned} 0 = &\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -2.5 & 6 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{8} \end{pmatrix} + \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{8} \end{pmatrix} \begin{pmatrix} -2.5 & 0 \\ 6 & -1 \end{pmatrix} \\ &- \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{8} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{8} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix} - \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix} = 0 \end{aligned}$$

and we have therefore showed that the given P-matrix relates to the state representation $X = \begin{pmatrix} x_1 \\ v_1 \end{pmatrix}$, for which we have known A, B and C matrices. If the state representation is inverted, i.e. $X = \begin{pmatrix} v_1 \\ x_1 \end{pmatrix}$, the Riccati equation will not hold and it can be realized that the only other state representation is the aforementioned one, for which we can show as above that the Riccati equation holds.

Using the C matrix given by this state representation, we can then calculate K using the (simplified) expression

$$K = PC^T = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}.$$

6. The third order system

$$\dot{x} = \begin{pmatrix} -0.5 & 1 & 1 \\ 0 & -1 & 1.5 \\ 0 & 0 & -2 \end{pmatrix} x + \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 3 \end{pmatrix} u$$
$$y = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} x$$

has the balanced realization

$$\dot{\xi} = \begin{pmatrix} -0.3658 & 0.2886 & -0.1586 \\ -0.2918 & -0.6964 & -0.3465 \\ 0.2358 & 1.542 & -2.438 \end{pmatrix} \xi + \begin{pmatrix} -0.8834 & -2.183 \\ -0.1861 & -0.8637 \\ -0.508 & 1.005 \end{pmatrix} u$$
$$y = \begin{pmatrix} -0.4974 & -0.01941 & -1.096 \\ -2.301 & 0.8833 & -0.2585 \end{pmatrix} \xi$$

with the observability gramian

$$O_\xi = \begin{pmatrix} 7.5775 & 0 & 0 \\ 0 & 0.5604 & 0 \\ 0 & 0 & 0.2603 \end{pmatrix}.$$

a. What is the controllability gramian S_ξ for the balanced realization? (0.5 p)

b. Reduce the system to a first order system without changing the behavior in stationarity. (2 p)

c. Calculate an error bound for the reduced system. (0.5 p)

Solution

a. In the balanced realization the observability gramian and controllability gramian are equal so $S_\xi = O_\xi$.

b. If we denote the state we want to keep ξ_1, and the states we want to reduce ξ_r, the system can be divided in the following way

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_r \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_r \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u$$
$$y = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_r \end{pmatrix}.$$

To keep the behavior in stationarity unchanged we want $\dot{\xi}_r = 0$ which gives an expression for ξ_r, namely that

$$\xi_r = A_{22}^{-1}(-A_{21}\xi_1 - B_2 u).$$

By inserting that instead of ξ_r in the reduced system we get

$$\begin{aligned} \dot{\xi}_1 &= (A_{11} - A_{12}A_{22}^{-1}A_{21})\xi_1 + (B_1 - A_{12}A_{22}^{-1}B_2)u \\ y &= (C_1 - C_2A_{22}^{-1}A_{21})\xi_1 - C_2A_{22}^{-1}B_2 u \end{aligned}.$$

The matrices are given by

$$\begin{aligned} A_{11} &= -0.3658 \\ A_{12} &= \begin{pmatrix} 0.2886 & -0.1586 \end{pmatrix} \\ A_{21} &= \begin{pmatrix} -0.2918 & 0.2358 \end{pmatrix}^T \\ A_{22} &= \begin{pmatrix} -0.6964 & -0.3465 \\ 1.542 & -2.438 \end{pmatrix} \\ B_1 &= \begin{pmatrix} -0.8834 & -2.183 \end{pmatrix} \\ B_2 &= \begin{pmatrix} -0.1861 & -0.8637 \\ -0.508 & 1.005 \end{pmatrix} \\ C_1 &= \begin{pmatrix} -0.4974 & -2.301 \end{pmatrix}^T \\ C_2 &= \begin{pmatrix} -0.01941 & -1.096 \\ 0.8833 & -0.2585 \end{pmatrix}. \end{aligned}$$

By inserting the numerical values we get the system

$$\begin{aligned} \dot{\xi}_1 &= \begin{pmatrix} -0.4480 \end{pmatrix}\xi_1 + \begin{pmatrix} -0.8738 & -2.4554 \end{pmatrix} u \\ y &= \begin{pmatrix} -0.3502 \\ -2.5818 \end{pmatrix}\xi_1 + \begin{pmatrix} 0.3170 & 0.3316 \\ -0.0357 & -0.8979 \end{pmatrix} u. \end{aligned}$$

c. An error bound is given by

$$\frac{||y - y_r||_2}{||u||_2} \leq 2\sigma_{r+1} + ... + 2\sigma_n = 2 \cdot 0.5604 + 2 \cdot 0.2603 = 1.6414$$

7. Consider the system in Figure 3. C,P and Δ are SISO transfer functions.

a. Assuming that $T = 0$, use the small gain theorem to find a condition on Δ that gives stability of the closed loop system. (1 p)

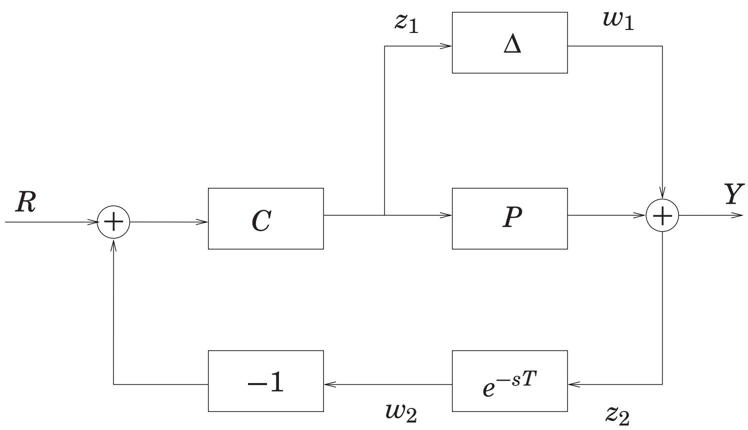

Figure 3 Block diagram for problem 7.

b. Assuming that $T = 1$, $\Delta = 0$ and $P = \frac{s+2}{s^2+s+1}$, calculate $C(s)$ using Internal Model Control if the controller for the un-delayed system would have been given by $C_0(s) = \frac{s+2}{s+3}$. (2 p)

c. For the controller above, how much can the time delay deviate from $T = 1$ without making the closed loop unstable? (1 p)

Solution

a. In order to use the small gain theorem we must obtain the transfer function G from w_1 to z_1. We have

$$z_1 = C(R - w_1 - Pz_1) \Rightarrow G_{w_1 \to z_1} = \frac{-C}{1 + PC}.$$

The upper bound on Δ is then

$$\|\Delta\|_\infty \leq \frac{1}{\|G_{w_1 \to z_1}\|_\infty}.$$

b. The rule of thumb says that we obtain Q as if there is no delay and then obtain the controller as

$$C(s) = \frac{Q(s)}{1 - Q(s)P(s)e^{-s}}$$

Given the controller to the model without delay $C_0 = \frac{s+2}{s+3}$ and the model without delay $P = \frac{s+2}{s^2+s+1}$, we calculate Q

$$Q(s) = \frac{C_0}{1 + C_0 P} = \frac{s^3 + 3s^2 + 3s + 2}{s^3 + 5s^2 + 8s + 7}$$

Now the controller is described as

$$C(s) = \frac{C_0}{1 + (1 - \mathrm{e}^{-s})PC_0} = \frac{s^3 + 3s^2 + 3s + 2}{s^3 + 5s^2 + 8s + 7 - e^{-s}(s^2 + 4s + 4)}$$

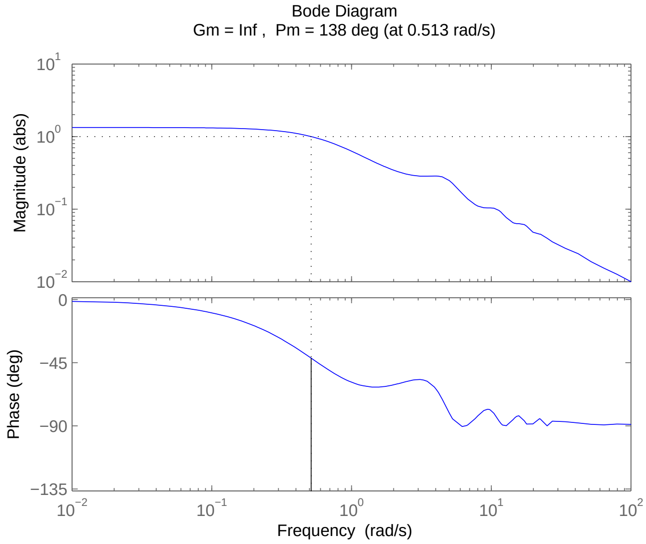

Figure 4 Bode diagram of PC for problem 7.

c. We can find the delay margin L_m for the open loop system $G_0 = PC$ by e.g. calculating (numerically) the crossover frequency ω_c from $|G_0(i\omega_c)| = 1$, the phase margin $\varphi_m = \pi - \arg(G(i\omega_c))$, and then use that $L_m = \dfrac{\varphi_m}{\omega_c}$. This gives $\omega_c = 0.5134$, $\varphi_m = 2.4128$, $L_m = 4.70$, so $T < 4.7$ will result in a stable closed loop system. The bode plot for PC is shown in Fig. 4.

The limit can also be seen by looking at the step response for the closed loop system

$$G_{cl} = \frac{PC_0}{1 + PC_0 + PC_0(\mathrm{e}^{-sT} - \mathrm{e}^{-s})}$$

where $T > 4.7$ gives an unstable response while $T < 4.7$ gives a stable one.