1.

a. The minors of the system are

$$\frac{s-1}{s+1},\quad \frac{9}{s+1},\quad \frac{1}{s+2},\quad \frac{1}{s+2},\quad \text{and}$$
$$\frac{s-1}{s+1}\cdot\frac{1}{s+2}-\frac{9}{s+1}\cdot\frac{1}{s+2}=\frac{s-10}{(s+1)(s+2)}$$

The least common denominator of these is

$$p(s)=(s+1)(s+2)$$

We thus have poles in -1 and -2, both stable.

The maximal minor of the system is the determinant, i.e. $\frac{s-10}{(s+1)(s+2)}$. Hence, the zero polynomial is $s-10$ and the system has a zero at 10.

The unstable zero gives a fundamental limitation on the rate of control. In particular, the specification

$$\left\|[I+G(i\omega)C(i\omega)]^{-1}\right\| < \frac{\sqrt{2}}{\sqrt{1+(10/\omega)^2}} \qquad \text{for all } \omega$$

would be impossible to satisfy with a stabilizing controller.

b. The transfer matrix is

$$\begin{aligned}
G(s) &= \begin{bmatrix} \frac{s-1}{s+1} & \frac{9}{s+1} \\ \frac{1}{s+2} & \frac{1}{s+2} \end{bmatrix} = \begin{bmatrix} 1-\frac{2}{s+1} & \frac{9}{s+1} \\ \frac{1}{s+2} & \frac{1}{s+2} \end{bmatrix} \\
&= \frac{1}{s+1}\begin{bmatrix} -2 & 9 \\ 0 & 0 \end{bmatrix} + \frac{1}{s+2}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\
&= \frac{1}{s+1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} -2 & 9 \end{bmatrix} + \frac{1}{s+2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

A state-space realization of the system is thus

$$\begin{aligned} \dot{x} &= Ax+Bu, \\ y &= Cx+Du, \end{aligned}$$

where

$$A=\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix},\quad B=\begin{bmatrix} -2 & 9 \\ 1 & 1 \end{bmatrix},$$
$$C=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\quad D=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

2. Calculate RGA for $G(iw)$ where $w = 2\pi f$.

$\text{RGA}(G(2\pi 0.1i) = \begin{bmatrix} -0.03 & 1.03 \\ 1.03 & -0.03 \end{bmatrix}$.

We thus see that we should control y_1 using u_2 and vice versa, and since the corresponding entry in the RGA matrix is very close to 1 we can expect good performance from this choice.

Solving this problem in Matlab could be done as

```
s=tf('s')
G = [1/(1+s) 10*s/(s+1); 10*s/(s+1) 1/(1+s)]
f=0.1
w=2*pi*f

Gw=evalfr(G,w)
rga=Gw.*(inv(Gw)')
```

3.

a.

$$z = \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} P_0 d + P_0 u \\ u \end{bmatrix}$$
$$y = x - r = P_0 d + P_0 u - r$$

$$\left(\frac{z}{y}\right) = \underbrace{\left(\begin{array}{cc|c} 0 & P_0 & P_0 \\ 0 & 0 & 1 \\ \hline -1 & P_0 & P_0 \end{array}\right)}_{\begin{bmatrix} P_{zw}(s) & P_{zu}(s) \\ P_{yw}(s) & P_{yu}(s) \end{bmatrix}} \left(\begin{array}{c} r \\ d \\ \hline u \end{array}\right)$$

b. The closed loop transfer function from w to z is $P_{zw} + P_{zu} Q P_{yw}$, where

$$Q(s) = \frac{C(s)}{1 + C(s) P_{yu}(s)} = \frac{C(s)}{1 + C(s) P_0(s)} \qquad C(s) = \frac{Q(s)}{1 - Q(s) P_0(s)}$$

The two controllers $C_1 = \frac{1}{s(s+1)}$ and $C_2 = 2$ give

$$Q_1(s) = \frac{s+1}{s^3 + 2s^2 + s + 1} \qquad Q_2(s) = \frac{s+1}{s+3}$$

respectively. All three specifications are convex in $Q(s)$, so the controller corresponding to $Q_3(s) = \frac{Q_1(s)+Q_2(s)}{2}$ must satisfy the specifications

$$\min_{t \in [5,\infty)} x(t) \geq \frac{0.94 + 0.68}{2} = 0.81$$
$$\max_{t \in [0,\infty)} u(t) \leq \frac{1.3 + 2}{2} \leq 1.65$$
$$\left\| \frac{P_0 C_3}{1 + P_0 C_3} \right\|_\infty \leq \frac{1.35 + 0.67}{2} \leq 1.01$$

which is good enough. The desired controller is

$$C_3(s) = \frac{Q_3(s)}{1 - Q_3(s)P_0(s)} = \frac{Q_1(s) + Q_2(s)}{2 - [Q_1(s) + Q_2(s)]P_0(s)}$$
$$= \frac{2s^3 + 4s^2 + 3s + 5}{2s^3 + 6s^2 + 4s + 1}$$

4.

a. The closed loop transfer function

$$\frac{K/(s+4)}{1 + K/(s+4)} = \frac{K}{s + 4 + K}$$

is stable if and only if $K > -4$.

b. The transfer function from n to v is

$$G(s) = \frac{K/(s+4)}{1 + K/(s+4)} \cdot \frac{2s}{s+4} = \frac{2Ks}{(s+4+K)(s+4)}$$

Hence for $K > -4$ (where G is stable)

$$|G|_\infty = \max_\omega \left| \frac{2Ki\omega}{16 + 4K - \omega^2 + (8+K)i\omega} \right|$$
$$= \max_\omega \frac{2\omega|K|}{\sqrt{(16 + 4K - \omega^2)^2 + \omega^2(8+K)^2}}$$
$$= \max_\omega \frac{2|K|}{\sqrt{(\frac{16+4K}{\omega^2} - 1)^2 + (8+K)^2}} = \left| \frac{2K}{8+K} \right|$$

According to the small gain theorem, the closed loop system is stable for every Δ with gain less than one provided that $|\frac{2K}{8+K}| < 1$. Equivalently

$$\begin{cases} 2K \le 8 + K \\ -2K \le 8 + K \end{cases}$$

or

$$-\frac{8}{3} \le K \le 8$$

5.

a. We need to solve

$$
\begin{aligned}
y^T Q y &= x^T C^T Q C x \implies \\
Q_1 &= C^T Q C = \begin{bmatrix} 10 & 10 \\ 10 & 10 \end{bmatrix} \\
R &= 1 \\
0 &= A^T S + SA + Q_1 - SBR^{-1}B^T S \\
L &= R^{-1}B^T S \\
\implies s_1 &= \frac{s_{12}^2}{2} - 5 \\
s_{12} &= s_{21} = -\frac{s_2^2}{2} + s2 + 5 \\
0 &= 2s_2 - \frac{(-\frac{s_2^2}{2} + s_2 + 5)^2}{2} - s_2^2 - s_2(-\frac{s_2^2}{2} + s_2 + 5) + 25
\end{aligned}
$$

The last equations has two real roots, which can be found by e.g plotting the graph. Since we need $s_2 > 0$ for S to be positive definite only the root in $s_2 = 6.9806$ is possible.

$$
S = \begin{bmatrix} 71.7 & -12.4 \\ -12.4 & 7.0 \end{bmatrix} \implies L = [\,-12.7 \quad 7.0\,]
$$

b.

$$
\begin{aligned}
E &= 1 \\
V &= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \\
0 &= AP + PA^T + V - PC^T E^{-1} PC^T \\
K &= PC^T E^{-1} \\
\implies 0 &= \begin{cases} 2p_1 - 2p_{12} - p_1(p_1 + p_{12}) - p_{12}(p_1 + p_{12}) + 2 \\ 2p_{12} - p_2 - p_1(p_2 + p_{12}) - p_{12}(p_2 + p_{12}) \\ 2p_2 - p_2(p_2 + p_{12}) - p_{12}(p_2 + p_{12}) + 1 \end{cases} \\
\implies p_2 &= \frac{2p_{12} - p_{12}(p_1 + p_{12})}{p_1 + p_{12} + 1} \\
p_1 &= 1 - p_{12} \pm \sqrt{3 - 4p_{12}} \\
0 &= \begin{cases} \dfrac{18p_{12} + 6p_{12}\sqrt{3 - 4p_{12}} - 12p_{12}^2\sqrt{3 - 4p_{12}} - 15p_{12}^2 + 4p_{12}^3 + 1}{(4p_{12} + 1)^2}, & p_1^{(+)} \\ \dfrac{18p_{12} - 6p_{12}\sqrt{3 - 4p_{12}} + 12p_{12}^2\sqrt{3 - 4p_{12}} - 15p_{12}^2 + 4p_{12}^3 + 1}{(4p_{12} + 1)^2}, & p_1^{(-)} \end{cases}
\end{aligned}
$$

We have two possible equations for p_{12} depending on which relation holds for p_1. We can immediately see that $p_{12} \leq 0.75$ for the solution to be real. Solving the equations numerically by eg plotting shows that only one case gives a real solution for p_{12} and that is $p_{12} = -29.84$

$$
P = \begin{bmatrix} 41.0 & -29.8 \\ -29.8 & 23.0 \end{bmatrix} \implies K = \begin{bmatrix} 12.1 \\ -6.9 \end{bmatrix}
$$

Solving this problem in Matlab could be done as

```
A=[1 -1; 0 1]
B=[0; 1]
C=[1 1]
R=1
Q=10
Qexp=C'*Q'*C

% LQ design
S=care(A, B, Qexp, R, zeros(2,1), eye(2))
L=inv(R)*B'*S

V = diag([2 1])
E = 1
% Kalman filter design
P=care(A', C', V, E, zeros(2,1), eye(2))
K = P*C'*inv(E)
```

6.

a. IV-D This has the quickest convergence since there is little measurement noise

III-C This has the slowest convergence due to the large uncertainty in the measurements

II-B Compared to A the x1 state converges quicker, which is due to the process model saying this state is more uncertain, and therefore the measurements influence it more than the x2 state

I-A Same reasoning as for II-B

b. Case III, since the filter doesn't put as much trust in the measurement it is of greater importance to have a correct model for the system behavior

c. Case IV, since the Kalman filter puts more trust in the measurement it is more sensitive to outliers

7.

a. The system consists of two independent SISO systems which are controllable and observable. The same is therefore true also for the MIMO system. Alternatively, this can be seen from the controllabilty Gramian and the observability Gramian to be computed in **b**, which both have full rank.

b. We find the controllabilty Gramian by solving the following equation

$$
\begin{aligned}
0 &= AS + SA^T + BB^T = \begin{bmatrix} -s_1 - s_1 + 1 & 0 \\ 0 & -2s_2 - 2s_2 + 0.1^2 \end{bmatrix} \\
\implies S &= \begin{bmatrix} 0.5 & 0 \\ 0 & 0.0025 \end{bmatrix}
\end{aligned}
$$

Similarly we find the observability Gramian by

$$
\begin{aligned}
0 &= A^T O + OA + C^T C = \begin{bmatrix} -o_1 - o_1 + 0.1^2 & 0 \\ 0 & -2o_2 - 2o_2 + 10^2 \end{bmatrix} \\
\implies O &= \begin{bmatrix} 0.005 & 0 \\ 0 & 25 \end{bmatrix}
\end{aligned}
$$

The hankel singular values are the square roots of the eigenvalues of $SO = \begin{bmatrix} 0.0025 & 0 \\ 0 & 0.0625 \end{bmatrix} \implies [0.05 \quad 0.25]$

c. We are looking for a transformation T such that $T^{-T}OT^{-1} = TST^T$. Since the Gramians are diagonal we attempt to find a diagonal transformation. $o_1/t_1^2 = 0.05 = s_1t_1^2, o_2/t_2^2 = 0.25 = s_2t_2^2$. We thus choose $t_1^2 = o_1/0.05 = 0.1 = 0.05/s_1, t_2^2 = o_2/0.25 = 100 = 0.25/s_2$

$$
\begin{aligned}
T &= \begin{bmatrix} \frac{1}{\sqrt{10}} & 0 \\ 0 & 10 \end{bmatrix}, \zeta = Tx \implies \\
A_\zeta &= TAT^-1 = A \\
B_\zeta &= TB = \begin{bmatrix} \frac{1}{\sqrt{10}} & 0 \\ 0 & 1 \end{bmatrix} \\
C_\zeta &= CT^{-1} = \begin{bmatrix} \frac{1}{\sqrt{10}} & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

Since the system consisted of two SISO systems this answer was also immediately obvious, the A-matrix won't change but the static gain in the system is evenly distributed between the B and C matrix.

d. We discard the first state, since it has the smallest Hankel singular value. Since there are no cross terms the reduced system simply becomes

$$
\begin{aligned}
\dot{\zeta}_2 &= -2\zeta_2 + [0 \quad 1]\, u \\
y &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \zeta_2 + \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix} u
\end{aligned}
$$

The direct term is a result from that we calculate the value of ζ_1 in stationarity and calculate y_1 using that.