FRTN10 Exercise 14. Old Exam Problems

14.1 Write down a state-space realization for the system

$$G(s) = \begin{pmatrix} \dfrac{1}{(s+1)(s+2)} & \dfrac{s+3}{(s+1)(s^2+6s+8)} \end{pmatrix}$$

14.2 A system has the transfer function $G(s) = \frac{1}{s+a}$, where $a > 0$. The input to the system is white noise with spectral density $\Phi_n = 1$. What is the spectral density of the output?

14.3 Steve is working with a process and he wants to design a controller for it. After identifying the transfer function $P(s)$ he decides to try a PI controller, $C_{PI}(s) = 0.88(1 + \frac{1}{s})$. With the PI controller, the disturbance response is very poorly damped, so he adds a lead filter. The controller is now:

$$C(s) = 0.88\left(1 + \frac{1}{s}\right)\frac{s/1.79 + 1}{s/8.94 + 1}.$$

The Bode diagrams of the controller and the loop transfer function can be seen in Figure 14.1. The disturbance response is still poorly damped. You realize that Steve has made a serious mistake when calculating his lead filter parameters.

a. What is Steve's mistake?

b. Improve the disturbance response by adjusting the lead filter (including the gain K). You have to follow the specifications:

- The cross-over frequency ω_c must not change
- The high frequency gain of the controller must not increase

14.4 A linear model of an inverted pendulum on a cart is given by

$$Y(s) = \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = G(s)U(s) = \begin{pmatrix} \dfrac{\omega_0^2}{s^2 - \omega_0^2} \\ \dfrac{1}{s} \end{pmatrix} U(s)$$

where Y_1 is the pendulum angle, Y_2 is the cart velocity and U (which is the control signal) is the acceleration of the cart. Consider the control system shown in Figure 14.2. The controller C_1 is used to stabilize the pendulum. Assume that a stabilizing controller has been designed and is given by

$$C_1(s) = \frac{B_1(s)}{A_1(s)}.$$

a. Let us now consider design of the cart velocity loop with input r and output y_2. In order to evaluate different control designs, it is useful to analyze the loop transfer function, $G_o(s) = C_2(s)G_{y_2m}$, where G_{y_2m} is the transfer function from m to y_2. Calculate G_{y_2m}.

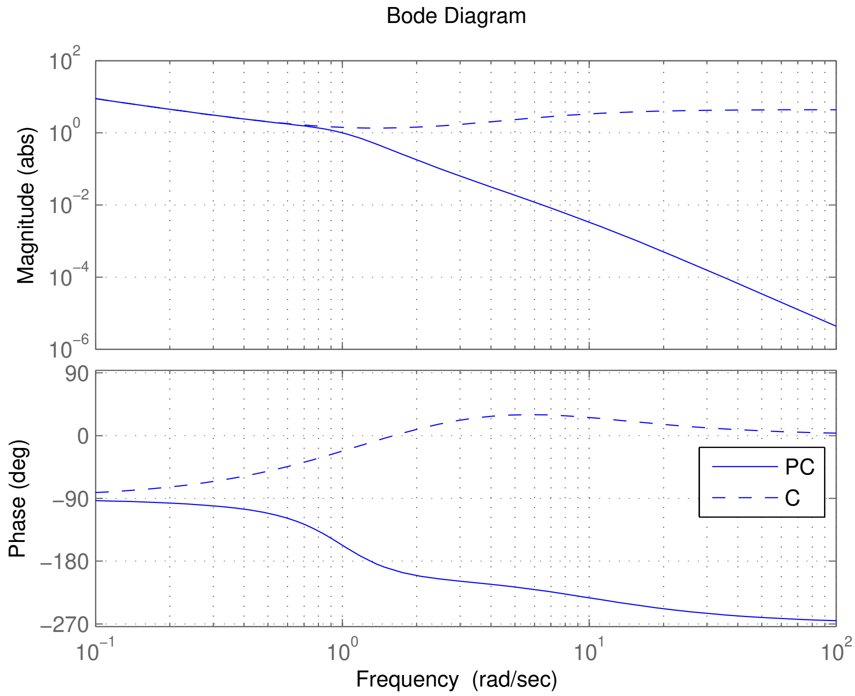

Figure 14.1 Loop transfer function PC and controller C in Problem 14.3

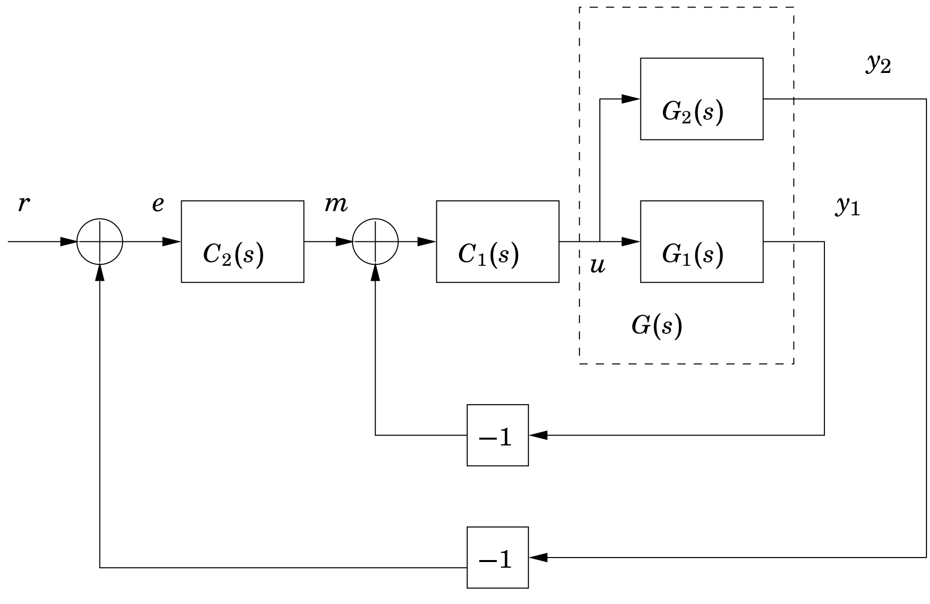

Figure 14.2 A block diagram for the inverted pendulum control system.

b. What can be said about performance limitations of the closed-loop system from r to y_2? *Notice:* You do not have to design any controllers!

c. Figure 14.3 shows four plots, where one of the plots shows the sensitivity and complementary sensitivity function of a closed-loop system discussed in **b**, with $\omega_0 = 1$ and a particular choise of $C_2(s)$. Which plot?

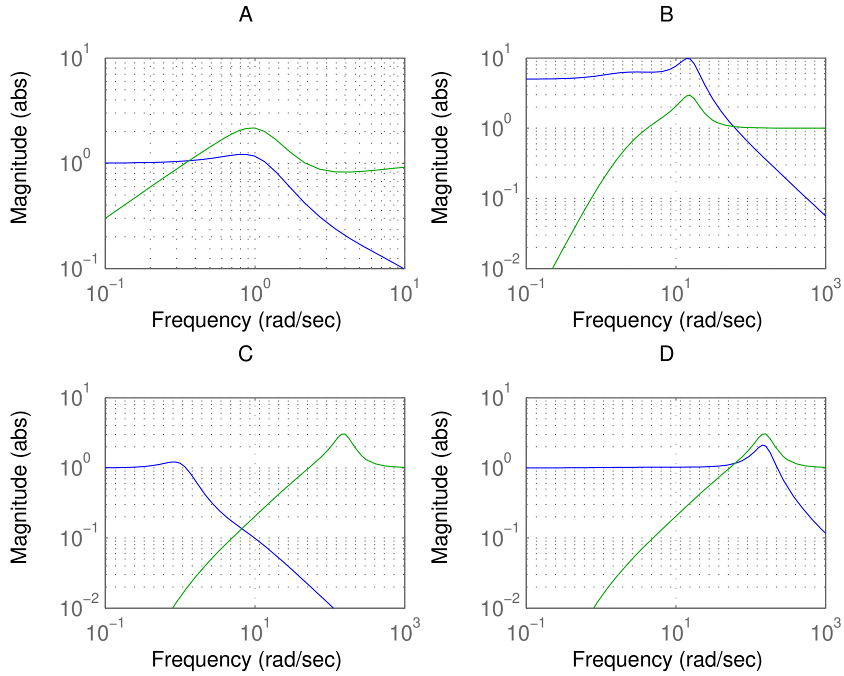

Figure 14.3 Magnitude plots of S and T in problem 6.

14.5 Solve the following problems:

a. Consider the system

$$G_2(s) = \begin{pmatrix} \dfrac{s-1}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \\ \dfrac{-6}{(s+1)(s+2)} & \dfrac{s-2}{(s+1)(s+2)} \end{pmatrix}.$$

Calculate the poles and zeros of $G_2(s)$. Are there any limitations on the achievable bandwidth?

b. Determine the RGA of $G_2(s)$ at $\omega = 0$ rad/s and choose reasonable input/output pairs for decentralized control. Can we expect decentralized control to work well for low frequencies?

14.6 Assume that we can model a physical process with the following transfer function

$$G(s) = \frac{(s+a)^m}{(s+b)^n},$$

where $m = 1 < n$ and $a, b > 0$. The IMC method was used to find a controller for this system, namely a PID controller with a lowpass filter

$$C(s) = K\frac{(1 + \frac{1}{T_i s} + T_d s)}{(s\frac{T_d}{N} + 1)}.$$

Determine what n the process must have and express K, T_i, T_d and N in a, b and the design parameter λ. What PID parameters are adjustable?

14.7 Recall the the quadruple tank system that was examined in Laboratory Exercise 2. The transfer function from the two inputs (u_1, u_2) to the two outputs (y_1, y_2) was

$$G(s) = \begin{pmatrix} \dfrac{\gamma_1 c_1}{1+sT_1} & \dfrac{(1-\gamma_2)c_1}{(1+sT_1)(1+sT_3)} \\[2ex] \dfrac{(1-\gamma_1)c_2}{(1+sT_2)(1+sT_4)} & \dfrac{\gamma_2 c_2}{1+sT_2} \end{pmatrix}$$

This time we will approach the problem by use of Q-optimization. The objective will be to keep the control errors $(e_1 = r_1 - y_1,\ e_2 = r_2 - y_2)$ low. For this reason we will choose r_1 as well as r_2 to be exogenous inputs w (see Figure 14.4) while e_1 and e_2 will be our exogenous outputs z. The signals r_1, r_2, y_1 and y_2 will all be given to the controller. Hence

$$z = \begin{bmatrix} r_1 - y_1 \\ r_2 - y_2 \end{bmatrix} \qquad w = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ r_1 \\ r_2 \end{bmatrix}$$

Determine the transfer function matrix

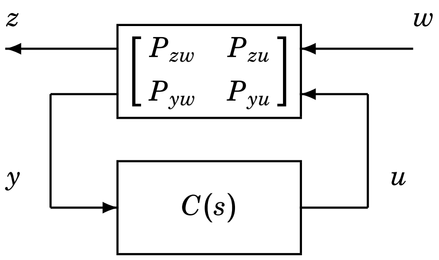

Figure 14.4 General form of a closed-loop system.

$$P = \begin{pmatrix} P_{zw} & P_{zu} \\ P_{yw} & P_{yu} \end{pmatrix}$$

(see Figure 14.4) for the quadruple tank system.

14.8 A MIMO system is described by the following transfer functions:

$$P(s) = \begin{bmatrix} \dfrac{1}{s+2} & -\dfrac{1}{s+2} \\[2ex] \dfrac{1}{s+1} & \dfrac{1}{s+4} \end{bmatrix}$$

a. Calculate the zero(s) of the process.

b. Suppose that we want to control the process by selecting input and output pairs for two SISO loops. How should we pair the inputs and outputs?

14.9 Consider the following process:

$$G(s) = \frac{4.2}{s^2 + 0.12s + 1}.$$

The control structure chosen is state-feedback design by minimizing the cost function

$$J = \int_0^\infty \left(y^T Q_1 y + u^T Q_2 u \right) dt$$

and a feedforward gain L_r such that we have the control signal

$$u(t) = -Lx(t) + L_r r(t).$$

Four different cost functions have been used and step responses from r to y have been plotted for performance comparison. However, the plots have not come in the correct order. Help the designer by pairing the correct weights below and step responses in Figure 14.5. A cost function might suit several step responses, give all alternatives!

A. $Q_1 = 1,\ Q_2 = 10$

B. $Q_1 = 1,\ Q_2 = 0.01$

C. $Q_1 = 100,\ Q_2 = 1000$

D. $Q_1 = 1,\ Q_2 = 100$

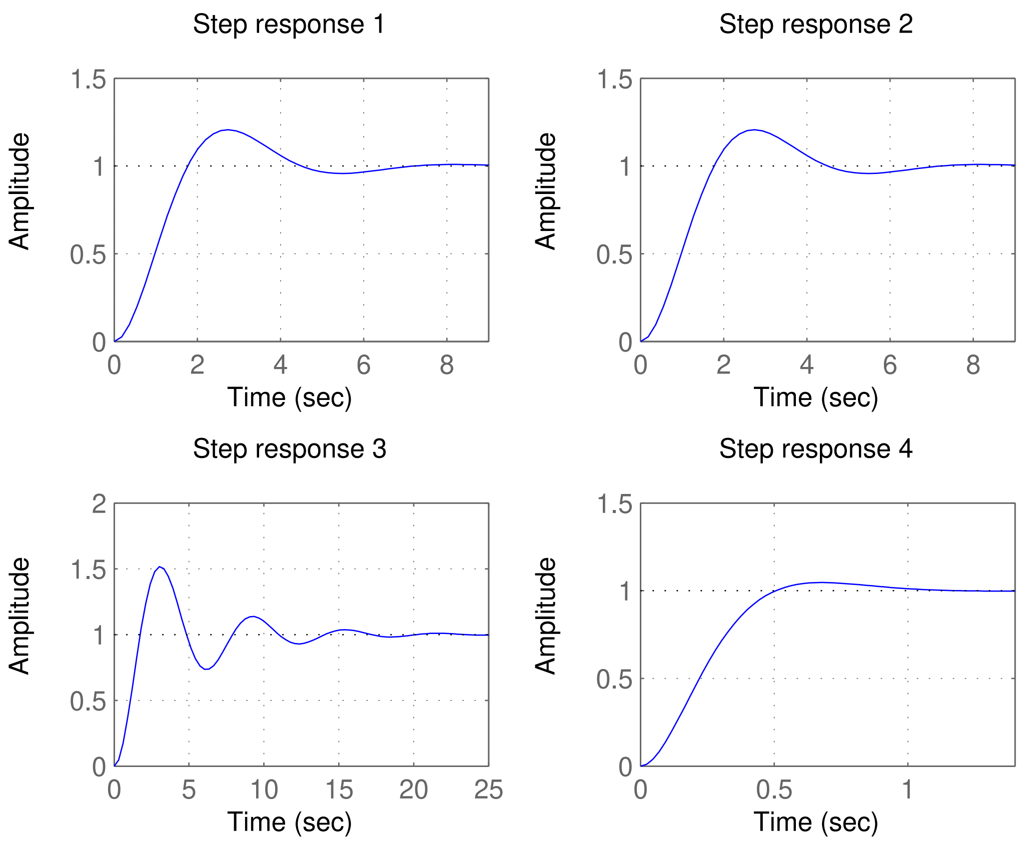

Figure 14.5 Step responses for Problem 14.9

Solutions to Exercise 14. Old Exam Problems

14.1 Partial fraction expansion gives

$$G(s) = \begin{bmatrix} \frac{1}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s^2+6s+8)} \end{bmatrix}$$
$$= \frac{1}{s+1}\begin{bmatrix} 1 & \frac{2}{3} \end{bmatrix} + \frac{1}{s+2}\begin{bmatrix} -1 & -\frac{1}{2} \end{bmatrix} + \frac{1}{s+4}\begin{bmatrix} 0 & -\frac{1}{6} \end{bmatrix}$$

so a realization in diagonal form can be written as

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix} x + \begin{bmatrix} 1 & \frac{2}{3} \\ -1 & -\frac{1}{2} \\ 0 & -\frac{1}{6} \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x$$

14.2 Denote the output by z. The spectral density of z is then

$$\Phi_z = |G(i\omega)|^2 \Phi_n(\omega) = \frac{1}{i\omega + a} \frac{1}{-i\omega + a} =$$
$$= \frac{1}{a^2 + \omega^2}$$

14.3 a. The bad damping in the disturbance response is a symptom of low phase margin, which is approximately 20° at $\omega_c = 1$ (as seen in the Bode diagram). The lead filter improves the phase margin, but the phase peak is located between the zero and pole at

$$\omega_p = \sqrt{1.79 \cdot 8.94} = 4\,\text{rad/s},$$

which is far from ω_c!

b. One way to improve the control is to move the phase peak to $\omega_c = 1$ by dividing the pole and zero by 4. The new controller is

$$C'(s) = K\left(1 + \frac{1}{s}\right)\frac{s/0.45 + 1}{s/2.24 + 1}.$$

The gain K should be chosen so that the cross-over frequency is preserved, that is $|C'(i\omega_c)| = |C(i\omega_c)|$, which gives $K = 0.45$.

The new controller gives an increase in phase margin of 19°. The high frequency gain

$$\lim_{s \to \infty} C(s)$$

actually decreases from 4.39 to 2.24.

14.4 a. Block scheme calculations gives

$$G_{y_2m} = \frac{G_2C_1}{1+G_1C_1} = \frac{B_1(s)(s^2-\omega_0^2)}{s[A_1(s)(s^2-\omega_0^2)+B_1(s)\omega_0^2]}$$

Note that this transfer function can be considered as the process in the outer loop.

b. A process zero at $z = \omega_0$ imposes a constraint on the achievable bandwidth of the closed-loop system—it is not possible to achieve a bandwidth significantly larger than ω_0.

c. Plot D shows too high bandwidth of the closed loop to be feasible. Plot B and C does not fulfill the constraint $S+T=1$. Plot A shows a bandwidth of about 1 rad/s which is reasonable—hence plot A.

14.5 a. The minors of size 1×1 are given by

$$\frac{s-1}{(s+1)(s+2)},\quad \frac{s}{(s+1)(s+2)},\quad \frac{-6}{(s+1)(s+2)},\quad \frac{s-2}{(s+1)(s+2)}$$

and the minor of size 2×2 is

$$\det(G_2(s)) = \frac{1}{(s+1)(s+2)}$$

with the least common denominator $p(s) = (s+1)(s+2)$. Thus, the system has poles in -1, -2 and no zeros. Since all poles and zeros are in the left half-plane, there are no fundamental limitations on the system bandwidth.

b.

$$\text{RGA} = G_2(0).*(G_2^T(0))^{-1} = \begin{pmatrix} -0.5 & 0 \\ -3 & -1 \end{pmatrix} .* \begin{pmatrix} -2 & 6 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Since the RGA is the identity matrix we can expect the system to be easily controlled with decentralized control at low frequencies. The identity matrix also gives us that it is suitable to pair input 1 with output 1 and input 2 with output 2.

14.6 Using the IMC method, we set $Q(s)$ to

$$Q(s) = \frac{1}{(\lambda s+1)^{n-m}}G^{-1}(s)$$

giving us the controller

$$C(s) = (1-Q(s)G(s))^{-1}Q(s) = \left(1-\frac{1}{(\lambda s+1)^{n-1}}\right)^{-1}\frac{(s+b)^n}{(s+a)(\lambda s+1)^{n-1}}$$
$$= \frac{(s+b)^n}{(s+a)((\lambda s+1)^{n-1}-1)}$$

To match the structure of the PID controller

$$\frac{K}{sT_i}\frac{(T_iT_ds^2+T_is+1)}{(s\frac{T_d}{N}+1)}$$

we see that we will need to choose $n=2$. This leaves us with

$$C(s)=\frac{b^2}{s\lambda a}\frac{(\frac{1}{b^2}s^2+\frac{2}{b}s+1)}{(\frac{s}{a}+1)},$$

such that we can now determine the PID parameters one by one

$$T_i=\frac{2}{b},\quad T_d=\frac{1}{T_ib^2}=\frac{1}{2b},\quad K=\frac{b^2T_i}{\lambda a}=\frac{2b}{\lambda a},\quad N=T_da=\frac{a}{2b}.$$

Since only K depends on λ, this is the only PID parameter that we have the possibility to tune ourselves.

P can be said to consist of several submatrices

$$P=\begin{pmatrix} P_{zw} & P_{zu}\\ P_{yw} & P_{yu}\end{pmatrix},$$

where

$$P_{zw}=\begin{pmatrix} P_{e_1r_1} & P_{e_1r_2}\\ P_{e_2r_1} & P_{e_2r_2}\end{pmatrix},\quad P_{zu}=\begin{pmatrix} P_{e_1u_1} & P_{e_1u_2}\\ P_{e_2u_1} & P_{e_2u_2}\end{pmatrix},$$
$$P_{yw}=\begin{pmatrix} P_{y_1r_1} & P_{y_1r_2}\\ P_{y_2r_1} & P_{y_2r_2}\\ P_{r_1r_1} & P_{r_1r_2}\\ P_{r_2r_1} & P_{r_2r_2}\end{pmatrix},\quad P_{yu}=\begin{pmatrix} P_{y_1u_1} & P_{y_1u_2}\\ P_{y_2u_1} & P_{y_2u_2}\\ P_{r_1u_1} & P_{r_1u_2}\\ P_{r_2u_1} & P_{r_2u_2}\end{pmatrix}$$

We can now determine all transfer functions that make up P:

$$P_{zw}=\begin{pmatrix}1&0\\0&1\end{pmatrix}=I,\quad P_{zu}=\begin{pmatrix}-P^0_{11} & -P^0_{12}\\ -P^0_{21} & -P^0_{22}\end{pmatrix}=-P^0,$$
$$P_{yw}=\begin{pmatrix}0&0\\0&0\\1&0\\0&1\end{pmatrix}=\begin{pmatrix}0\\I\end{pmatrix},\quad P_{yu}=\begin{pmatrix}P^0_{11} & P^0_{12}\\ P^0_{21} & P^0_{22}\\ 0&0\\0&0\end{pmatrix}=\begin{pmatrix}P^0\\0\end{pmatrix}$$

14.8 a. The minors are

$$\frac{1}{s+2},\ -\frac{1}{s+2},\ \frac{1}{s+1},\ \frac{1}{s+4},\ \det P=\frac{2s+5}{(s+4)(s+2)(s+1)}$$

with the least common denominator $p(s)=(s+1)(s+2)(s+4)$. The maximal minor is

$$\det P(s)=\frac{2s+5}{(s+4)(s+2)(s+1)}$$

which already has $p(s)$ as denominator. The zeros are given by $2s+5=0$, yielding $s=-2.5$.

14.8 b. Here RGA(0) is calculated.

$$G(0) = \begin{bmatrix} 0.5 & -0.5 \\ 1 & 0.25 \end{bmatrix}$$

Then RGA(0) becomes

$$\text{RGA}(0) = G(0).*(G(0)^{-1})^T = \begin{bmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{bmatrix}$$

and thus output 1 should pair with input 2 and output 2 with input 1.

14.9 We see that A and C will give the same controller since we have just scaled the weights by 100, so A and C will correspond to Step response 1 and 2. Notice that the system is very oscillative. D has much larger weight on the control signal, thus we will not be able to get a fast system that dampens the oscillative system, hence D must correspond to Step response 3. B will correspond to Step response 4. We have very small weight on the control signal compared to output, which will give a fast system.