FRTN10 Exercise 11. Youla Parametrization, Internal Model Control

11.1 Consider the control system in Figure 11.1, designed for the stable, linear, SISO system P_0. We first want to rewrite the system into the general form in Figure 11.2. In this figure, w are the external inputs to the system (e.g., disturbances and reference signals), z gathers all signals that we are interested in controlling, u are the output signals from C, and y contains all signals used by the controller (e.g., reference signals and measurements).

a. Let

$$w = \begin{pmatrix} d \\ n \end{pmatrix}, \quad z = \begin{pmatrix} x \\ v \end{pmatrix}$$

The generalized plant P in Figure 11.2 consists of four subsystems:

$$P = \begin{pmatrix} P_{zw} & P_{zu} \\ P_{yw} & P_{yu} \end{pmatrix}$$

Determine the transfer matrices P_{zw}, P_{zu}, P_{yw} and P_{yu}.

b. The closed-loop system from w to z is denoted G_{zw}. Show that $G_{zw} = P_{zw} + P_{zu}C(I - P_{yu}C)^{-1}P_{yw}$ for the general system in Figure 11.2.

c. Determine G_{zw} for the system in Figure 11.1. Introduce the Youla parameter $Q = \frac{C}{1-P_{yu}C}$ and show that each element of G_{zw} is an affine function of Q.

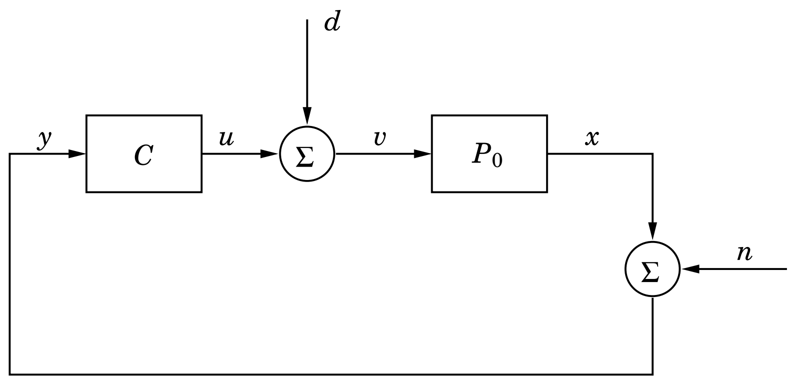

Figure 11.1 Original block diagram in Problem 11.1.

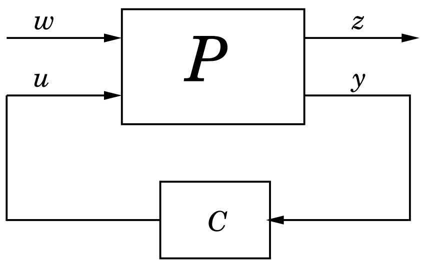

Figure 11.2 General closed-loop system in Problems 11.1 and 11.2.

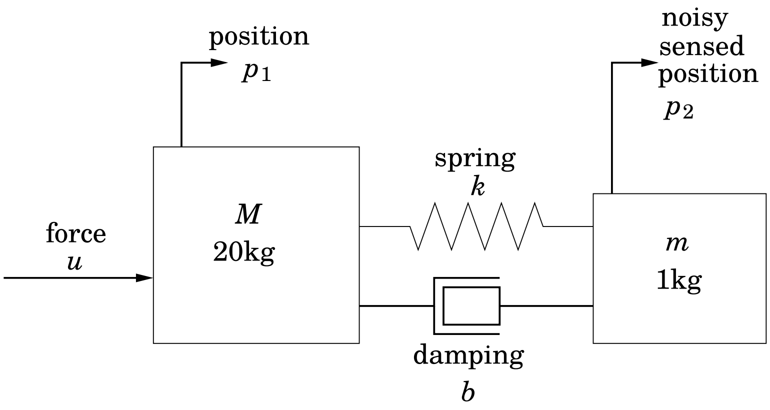

Figure 11.3 Mass spring system in Problem 11.2.

11.2 *Note: You should have solved this problem before you start working on Exercise 12.*

Let us consider the physical system shown in Figure 11.3, showing two masses, lightly coupled through a spring with spring constant k and damping b. The only sensor signal we have is the noisy measurement p_2+n of the position, p_2, for the small mass, m. The purpose of the controller is to make the position of the large mass, p_1, follow a reference input, r, such that the control error e becomes small. This is in turn weighted against controller effort in a quadratic cost function (the objective), expressed as

$$J = \int_0^\infty \left(\gamma e^2(t) + \rho u^2(t)\right) dt$$

Minimization of this function will be subject to constraints on:

- the magnitude of the control signal $|u(t)| < u_{max}$ (the force acting on the large mass) during a reference step
- step response overshoot, rise time and settling time from r to the position p_1 (performance constraint)
- the maximum norm of the sensitivity function, $\|S(i\omega)\|_\infty \leq M_s$ (robustness constraint)

The system can be described by the equations of motion

$$\begin{aligned} M\ddot{p}_1 + b(\dot{p}_1 - \dot{p}_2) + k(p_1 - p_2) &= u \\ m\ddot{p}_2 + b(\dot{p}_2 - \dot{p}_1) + k(p_2 - p_1) &= 0 \end{aligned}$$

Introducing the state vector

$$x = \begin{pmatrix} \dot{p}_1 \\ p_1 \\ \dot{p}_2 \\ p_2 \end{pmatrix}$$

we can write the system in state-space form as

$$\begin{aligned} \dot{x} &= Ax + Bu \\ p_1 &= C_1 x \\ p_2 &= C_2 x \end{aligned}$$

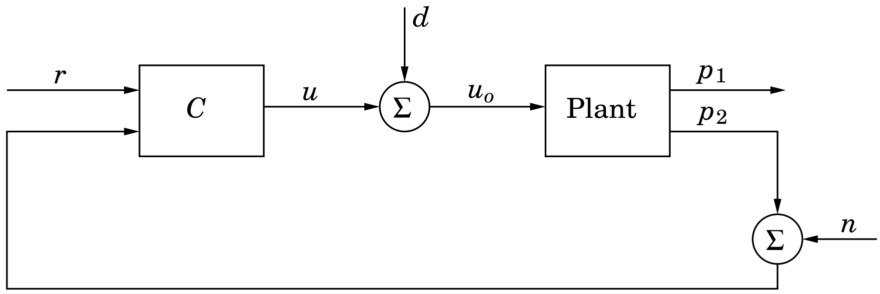

Figure 11.4 The block diagram of the control system in Problem 11.2.

where

$$A = \begin{pmatrix} -b/M & -k/M & b/M & k/M \\ 1 & 0 & 0 & 0 \\ b/m & k/m & -b/m & -k/m \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1/M \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$$

Let $M = 20$ kg, $m = 1$ kg, $k = 32$ N/m and $b = 0.3$ Ns/m.

Now, consider the problem to set up this system in a form such that we can optimize over the Q parametrization. Have Figure 11.4 as a reference. Then, the exogenous inputs, w, of the system are

- the reference r,
- the measurement noise n,
- the input disturbance d.

The performance outputs, z, are

- the position p_1 of the large mass M,
- the input to the plant, u_o,
- the control error $e = r - p_1$.

The control signal, u, to the plant is

- the force u on the large mass M.

The controller inputs y are

- the reference r,
- the noisy measurement $p_2 + n$.

In other words, we have

$$w = \begin{pmatrix} r \\ n \\ d \end{pmatrix}, \quad z = \begin{pmatrix} p_1 \\ u_o \\ e \end{pmatrix}, \quad y = \begin{pmatrix} r \\ p_2 + n \end{pmatrix}.$$

With these variables, we can rewrite the system in the general form shown in Figure 11.2. In state-space form, this becomes

$$\dot{x} = Ax + B_w w + Bu \tag{11.1}$$

$$z = C_z x + D_{zw} w + D_{zu} u \tag{11.2}$$

$$y = C_y x + D_{yw} w + D_{yu} u \tag{11.3}$$

a. Determine all matrices in equations (11.1)–(11.3).

b. In the next exercise session we will use software that solves the minimization problem. This software will need to know the general process P, determined by (11.1)–(11.3), and the element indices of the closed-loop transfer function G_{zw} corresponding to the constraints and cost function specified for the control design problem. For instance, the step response overshoot, rise time and settling time will correspond to $G_{p_1 r}$ which has index $(1, 1)$. Determine the rest of these indices.

c. Introducing the Youla parameter $Q = C(I - P_{yu}C)^{-1}$, how many inputs and outputs will Q have?

11.3 Derive a controller using the IMC method for the following system:

$$P(s) = \frac{6 - 3s}{s^2 + 5s + 6}.$$

Place any extra closed-loop poles in $-1/\lambda$. Show that the controller has the form of a PID controller with a first-order measurement filter, i.e.,

$$K\left(1 + \frac{1}{T_i s} + T_d s\right)\frac{1}{sT + 1}$$

11.4 Processes in industry often have time delays that limit the achievable performance. Consider the simple process

$$P(s) = \frac{1}{s+1} e^{-4s},$$

which is a delay dominant process (i.e., the time delay is longer than the time constant).

a. Use IMC to design a delay-compensating controller, using the approach of ignoring the time delay when $Q(s)$ is calculated.

b. Use IMC to design a delay-compensating controller, using the approach of approximating the time delay with a first order Padé approximation,

$$e^{-sL} \approx \frac{1 - sL/2}{1 + sL/2},$$

and then ignoring the unstable zero when $Q(s)$ is calculated.

c. **(*)** For both cases above, draw the Nyquist plot for the loop transfer function when $\lambda = 3$ and conclude whether the closed-loop system is stable.

Solutions to Exercise 11. Youla Parametrization, Internal Model Control

11.1 a. We can divide P even further into smaller parts such that

$$P_{zw} = \begin{pmatrix} P_{xd} & P_{xn} \\ P_{vd} & P_{vn} \end{pmatrix}, \quad P_{zu} = \begin{pmatrix} P_{xu} \\ P_{vu} \end{pmatrix}, \quad P_{yw} = \begin{pmatrix} P_{yd} & P_{yn} \end{pmatrix}$$

Looking at the block diagram of the closed-loop system and removing the controller C, we see that

$$\begin{aligned} &P_{xd} = P_0, \quad P_{xn} = 0, \quad P_{vd} = 1, \quad P_{vn} = 0 \\ &P_{xu} = P_0, \quad P_{vu} = 1 \\ &P_{yd} = P_0, \quad P_{yn} = 1 \\ &P_{yu} = P_0 \end{aligned}$$

This gives us the transfer matrix

$$P = \begin{pmatrix} P_0 & 0 & P_0 \\ 1 & 0 & 1 \\ P_0 & 1 & P_0 \end{pmatrix},$$

so that

$$P_{zw} = \begin{pmatrix} P_0 & 0 \\ 1 & 0 \end{pmatrix}, \quad P_{zu} = \begin{pmatrix} P_0 \\ 1 \end{pmatrix}, \quad P_{yw} = \begin{pmatrix} P_0 & 1 \end{pmatrix}, \quad P_{yu} = P_0$$

b.

$$\begin{aligned} &u = Cy \\ &y = P_{yu}u + P_{yw}w = P_{yu}Cy + P_{yw}w \quad \Rightarrow \quad y = (I - P_{yu}C)^{-1}P_{yw}w \\ &z = P_{zw}w + P_{zu}u = P_{zw}w + P_{zu}Cy = \left(P_{zw} + P_{zu}C(I - P_{yu}C)^{-1}P_{yw}\right)w \end{aligned}$$

c. Using the result from **b.** we get

$$\begin{aligned} G_{wz} &= \begin{pmatrix} P_0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} P_0 \\ 1 \end{pmatrix} C(1 - P_0C)^{-1} \begin{pmatrix} P_0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} P_0 & 0 \\ 1 & 0 \end{pmatrix} + \frac{C}{1 - P_0C} \begin{pmatrix} P_0^2 & P_0 \\ P_0 & 1 \end{pmatrix} \\ &= \frac{1}{1 - P_0C} \begin{pmatrix} (P_0 - P_0^2C) + P_0^2C & P_0C \\ (1 - P_0C) + P_0C & C \end{pmatrix} \end{aligned}$$

Introducing $Q = \frac{C}{1 - P_{yu}C}$ gives

$$G_{zw} = P_{zw} + P_{zu}QP_{yw} = \begin{pmatrix} P_0 & 0 \\ 1 & 0 \end{pmatrix} + Q \begin{pmatrix} P_0^2 & P_0 \\ P_0 & 1 \end{pmatrix} = \begin{pmatrix} P_0 + P_0^2Q & P_0Q \\ 1 + P_0Q & Q \end{pmatrix}$$

where each element of G_{zw} is affine in Q.

11.2 a. From the equation for the plant and the diagram of the control loop (removing the controller), we can see that the generalized plant is described by

$$\begin{aligned}
\dot{x} &= Ax + B(u+d) = Ax + \begin{pmatrix} 0 & 0 & B \end{pmatrix}\begin{pmatrix} r \\ n \\ d \end{pmatrix} + Bu \\
&= Ax + B_w w + Bu \\
z &= \begin{pmatrix} p_1 \\ u_o \\ e \end{pmatrix} = \begin{pmatrix} C_1 x \\ u+d \\ r-p_1 \end{pmatrix} = \begin{pmatrix} C_1 x \\ u+d \\ r - C_1 x \end{pmatrix} \\
&= \begin{pmatrix} C_1 \\ 0 \\ -C_1 \end{pmatrix} x + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} r \\ n \\ d \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u \\
&= C_z x + D_{zw} w + D_{zu} u \\
y &= \begin{pmatrix} r \\ p_2 + n \end{pmatrix} = \begin{pmatrix} r \\ C_2 x + n \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ C_2 \end{pmatrix} x + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} r \\ n \\ d \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} u \\
&= C_y x + D_{yw} w + D_{yu} u
\end{aligned}$$

b. The constraint on the control signal magnitude, $|u(t)| \leq u_{max}$, will correspond to the closed-loop transfer matrix $G_{u_o r}$, with index $(2,1)$. In the original block diagram, we see that the sensitivity function is given by the transfer function $G_{u_o d}$. The M_s constraint will therefore correspond to the index $(2,3)$. The objective function will be related to two indices, namely those associated with G_{er} and $G_{u_o r}$, $(3,1)$ and $(2,1)$.

c. Since P_{zu} is a 3×1 system and P_{yw} is a 2×3 system, Q must be 1×2. Therefore we have that $Q = [Q_1 \quad Q_2]$.

11.3 The system has a zero in the right half-plane. There are many ways to choose the Q filter for IMC; here we use the approach of replacing the unstable zero by its mirrored counterpart when forming Q. We also need to add a pole to $Q(s)$ to make it proper, which we place in $s = -\lambda^{-1}$. We get

$$Q(s) = \frac{s^2 + 5s + 6}{(6+3s)(\lambda s + 1)} = \frac{s+3}{3(\lambda s + 1)}$$

The controller becomes

$$C(s) = \frac{s^2 + 5s + 6}{s(3\lambda s + 6(\lambda + 1))}$$

which can be rewritten as

$$C(s) = \frac{5}{6(1+\lambda)}\left(1 + \frac{6}{5s} + \frac{s}{5}\right)\frac{1}{\frac{3\lambda}{6(\lambda+1)}s + 1}.$$

This corresponds to a PID controller in series with a lowpass filter.

11.4 a. We get

$$Q(s) = \frac{(P(s)e^{4s})^{-1}}{\lambda s + 1}$$

Hence, the controller is given by

$$C(s) = \frac{Q(s)}{1 - Q(s)P(s)} = \frac{s+1}{\lambda s + 1 - e^{-4s}}$$

b. When we calculate the $Q(s)$-transfer function, we exclude $1 - 2s$. Thus, we then have

$$Q(s) = \frac{(s+1)(2s+1)}{(\lambda s + 1)^2}.$$

Hence we have the controller

$$C(s) = \frac{Q(s)}{1 - Q(s)P(s)} = \frac{(s+1)(2s+1)}{(\lambda s + 1)^2 - (1 + 2s)e^{-4s}}$$

c. The Nyquist plots can be generated in Matlab, using the following lines of code. (Note that feedback delay systems are not always handled by Control System Toolbox.)

```
>> lambda = 3;
>> w = logspace(-2,2,1000);
>> P = 1./(1+i*w).*exp(-4*i*w);
>> Fy1 = (i*w+1)./(lambda*i*w+1-exp(-4*i*w));
>> Fy2 = (i*w+1).*(1+2i*w)./((lambda*i*w+1).*(lambda*i*w+1)-(1+2*i*w).*exp(-4*i*w));
>> figure
>> plot(P.*Fy1)
>> grid
>> figure
>> plot(P.*Fy2)
>> grid
```

From the plots (Figure 11.1) we see that neither encircles −1 and the closed-loop system is stable in both cases.

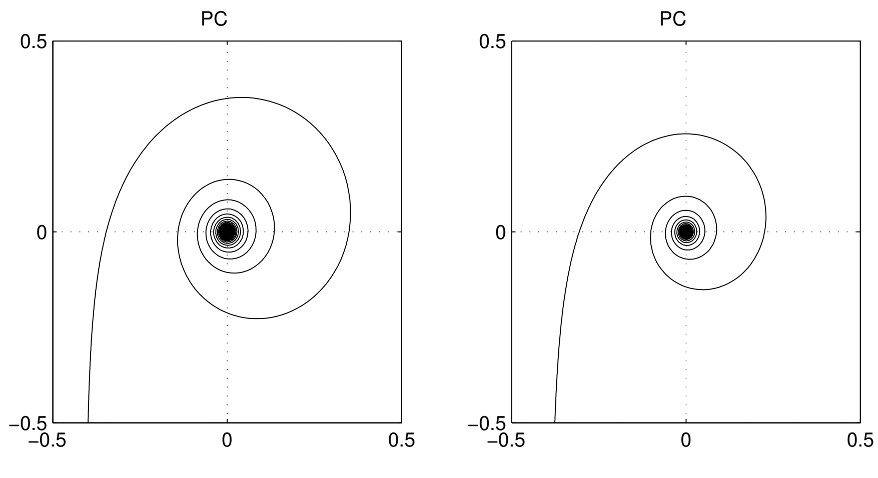

Figure 11.1 Nyquist plots of the loop transfer functions in Problem 11.4 **a** (left) and **b** (right).