FRTN10 Exercise 10. LQG, Preparations for Lab 3

10.1 Consider the system

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 6 \\ 0 & 4 \end{pmatrix} u + w_1$$
$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} x + w_2$$

where w_1 and w_2 are independent white noise processes. You should design an LQG controller that minimizes the following cost function:

$$J = \mathrm{E}\left((x_1 + x_2)^2 + u^T u\right)$$

a. Design the LQG controller in Matlab, initially assuming that the process and measurement noise have unit intensities.

Useful commands: `lqr, kalman, lqgreg`

b. Using the states x and $\hat{x}$, write the closed-loop system in state-space form using symbols. Use L for the state-feedback gain and K for the Kalman filter gain.

c. Simulate the system without noise from the initial state $x = \begin{pmatrix} 1 & -1 \end{pmatrix}^T$. Plot both process states and estimated states. The Kalman filter begins with its estimates in 0. Try some different values of R_2 (much smaller or much larger than the initial value) and repeat the design and simulation. What conclusions can you draw?

Useful commands: `lqgreg, feedback, initial`

10.2 Do the three preparatory exercises for Laboratory Session 3. The lab manual is found on the course homepage.

10.3(*) Consider the problem of controlling a double integrator

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + v_1$$

where the white noise v_1 has intensity I. We can only measure x_1, unfortunately with added white noise also of intensity 1. We want to minimize the cost function

$$J = \mathrm{E}\left(x_1^2 + x_2^2 + u^2\right)$$

Solve the control design problem by hand and give the LQG controller in state-space form.

Solutions to Exercise 10. LQG, Preparations for Lab 3

10.1 a. See the Matlab code in **c** below.

b. Using the state vector $x_e = \begin{pmatrix} x^T & \hat{x}^T \end{pmatrix}^T$ and the obvious notation A, B, C, we get the system

$$\dot{x}_e = \begin{pmatrix} A & -BL \\ KC & A-BL-KC \end{pmatrix} x_e + \begin{pmatrix} I \\ 0 \end{pmatrix} w_1 + \begin{pmatrix} 0 \\ K \end{pmatrix} w_2$$
$$z = \begin{pmatrix} C & 0 \end{pmatrix} x_e$$

c. With smaller R_2 the estimated states converge faster to the actual states, and the controlled output z converges faster to zero. The opposite holds for larger R_2. See Figures 10.1-10.2 and Matlab code below.

As shown in Exercise 9.1, only the relation between process noise and measurement noise matters. Increasing R_1 by some factor or decreasing R_2 by the same factor will thus have the same effect.

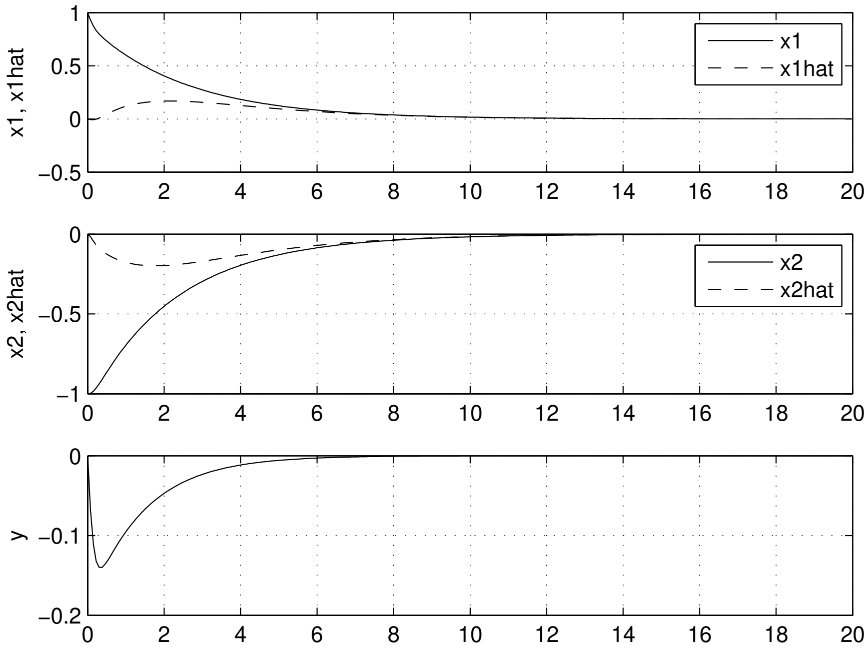

Figure 10.1 Initial response if R_2 10 times smaller.

```
A = [0 1; 0 0];
B = [1 6; 0 4];
C = [1 1];

% LQ design
process = ss(A,B,C,0);
Q1 = C'*C;
Q2 = eye(2);
```

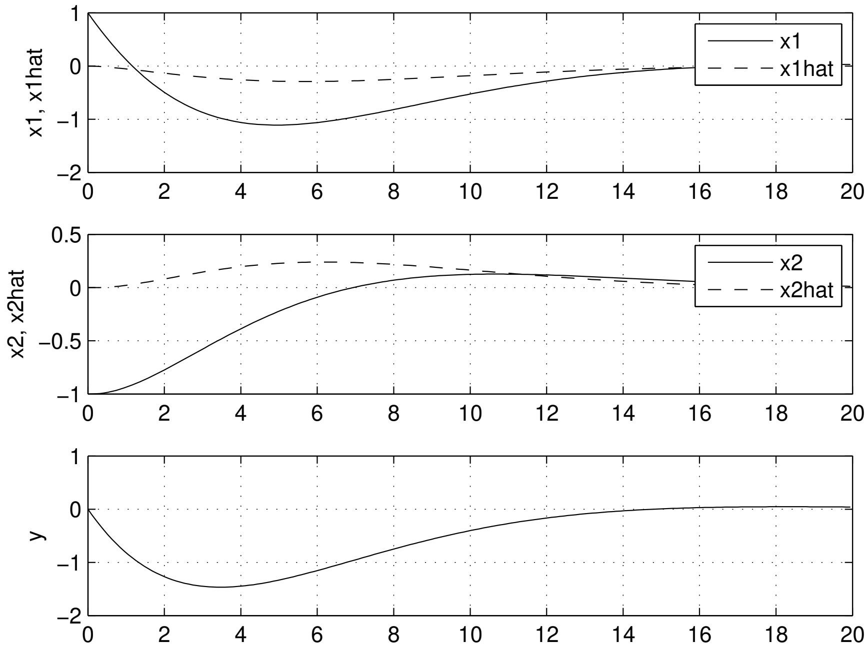

Figure 10.2 Initial response if R_2 100 times larger).

```
L = lqr(process,Q1,Q2);

% Kalman filter design
G = eye(2);
sysk = ss(A,[B G],C,0);
R1 = eye(2);
R2 = 1;
Kest = kalman(sysk,R1,R2);

% Construct regulator and form closed loop
reg = lqgreg(Kest,L);
closed_loop = feedback(process,-reg);

% Plot response
[Y,T,X] = initial(closed_loop,[1 -1 0 0],0:0.01:20);
subplot(311)
plot(T, X(:,1)); hold on; plot(T, X(:,3),'--'); grid
legend('x1','x1hat'); ylabel('x1, x1hat')
subplot(312)
plot(T, X(:,2)); hold on; plot(T, X(:,4),'--'); grid
legend('x2','x2hat'); ylabel('x2, x2hat')
subplot(313)
plot(T,Y); grid; ylabel('y');
```

10.2 No solutions provided.

10.3 To solve the problem we need to design an LQ state feedback and a Kalman filter. The problem parameters are given by

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q_2 = 1, \quad R_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R_2 = 1$$

For the LQ state feedback gain, we have to solve the Riccati equation

$$A^T S + SA + Q_1 - SBQ_2^{-1} B^T S = 0$$

with

$$S = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}$$

This gives the following equations,

$$\begin{aligned} 1 - s_2^2 &= 0 \\ s_1 - s_2 s_3 &= 0 \\ 2s_2 + 1 - s_3^2 &= 0 \end{aligned}$$

with the positive definite solution $s_1 = s_3 = \sqrt{3}$, $s_2 = 1$. This gives the state feedback vector $L = B^T S = \begin{pmatrix} 1 & \sqrt{3} \end{pmatrix}$.
For the Kalman filter we must solve the Riccati equation

$$AP + PA^T + R_1 - PC^T CP = 0$$

with

$$P = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}$$

yielding the equations

$$\begin{aligned} 2p_2 + 1 - p_1^2 &= 0 \\ p_3 - p_1 p_2 &= 0 \\ 1 - p_2^2 &= 0 \end{aligned}$$

Reusing the solution for S we have that $p_1 = p_3 = \sqrt{3}$ and $p_2 = 1$ and $K = PC^T = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$
The LQG controller is given by

$$\begin{aligned} \dot{\hat{x}} &= (A - BL - KC)\hat{x} + Ky \\ u &= -L\hat{x} \end{aligned}$$

and we have that

$$A - BL - KC = \begin{pmatrix} -\sqrt{3} & 1 \\ -2 & -\sqrt{3} \end{pmatrix}$$