FRTN10 Exercise 9. Kalman Filtering

9.1 Consider the unstable first-order system

$$\begin{aligned} \dot{x}(t) &= x(t) + u(t) + w_1(t) \\ y(t) &= x(t) + w_2(t) \end{aligned}$$

The uncorrelated noise signals $w_i(t)$ are white with intensities R_i. We want to investigate how the optimal Kalman filter depends on noise parameters.

a. Show that the Kalman filter gain only depends on the ratio $\beta = R_1/R_2$.

b. Find the observer error dynamics, i.e., the dynamics of the estimation error $\tilde{x}(t) = x(t) - \hat{x}(t)$.

c. How does the error dynamics depend on the ratio $\beta = R_1/R_2$? Interpret the result for large β (process noise much larger than measurement noise), and for small β (measurement noise much larger than process noise).

9.2 A Kalman filter should be designed for the second-order system

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) + w_1(t) \\ y(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x(t) + w_2(t) \end{aligned}$$

where w_1 and w_2 are uncorrelated white noise processes with intensities $R_1 = \left(\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right)$ and $R_2 = 1$, respectively.

a. Calculate the minimum observer error covariance P and the optimal Kalman filter gain K.

b. Write down the resulting filter equations for $\hat{x}_1$ and $\hat{x}_2$.

c. **(*)** Find the minimum error covariance P and the optimal filter gain K using `lqe` in Matlab.

9.3 Consider an integrator process driven by unit intensity white noise:

$$\dot{x}(t) = w_1(t), \qquad R_1 = 1$$

For each of the cases below, assuming that an optimal Kalman filter should be designed, compute the minimum observer error variance.

a. There is one noisy measurement of x, given by

$$y(t) = x(t) + w_2(t), \qquad R_2 = 1$$

b. There are two independent noisy measurements of x, given by

$$\begin{aligned} y_1(t) &= x(t) + w_{21}(t), \qquad & R_{21} &= 1 \\ y_2(t) &= x(t) + w_{22}(t), \qquad & R_{22} &= 10 \end{aligned}$$

c. There are two dependent noisy measurements of x, given by

$$\begin{aligned} y_1(t) &= x(t) + w_{21}(t), \\ y_2(t) &= x(t) + w_{22}(t), \end{aligned} \qquad R_2 = \begin{pmatrix} 1 & 1 \\ 1 & 10 \end{pmatrix}$$

9.4 We would like to design an output feedback controller for the stable second-order system

$$G(s) = \frac{1}{(s+1)^2}.$$

The first step is to design a Kalman filter. The process and its disturbances have been modeled as shown in Figure 9.1, where w_{11}, w_{12} and w_2 are uncorrelated, unit intensity white noise processes. A low-frequency input disturbance has been modeled by filtering w_{12} through a low-pass filter $\frac{1}{s+\epsilon}$, where $\epsilon > 0$ is a small number.

Write down the system in state-space form and find all the relevant matrices needed to state a Riccati equation for the Kalman filter.

9.5 Consider control of a DC-motor,

$$G(s) = \frac{1}{s(s+1)}$$

Introduce the state variables $x_1 = y$, $x_2 = \dot{y}$. White process noise is active on both states with intensity 1 and with input vector $\begin{pmatrix} 0.1 & 0.1 \end{pmatrix}^T$. There is also noise on the measurements with intensity 0.1. This gives the following state-space model

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) + \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix} v_1(t) \\ y(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x(t) + v_2(t) \end{aligned}$$

with $R_1 = 1$, $R_2 = 0.1$ and $R_{12} = 0$

a. The motor will be connected to an external system that might be oscillatory around the frequency 0.5 rad/s, but there is no detailed knowledge about its properties. In order not to excite the oscillatory modes we would like the

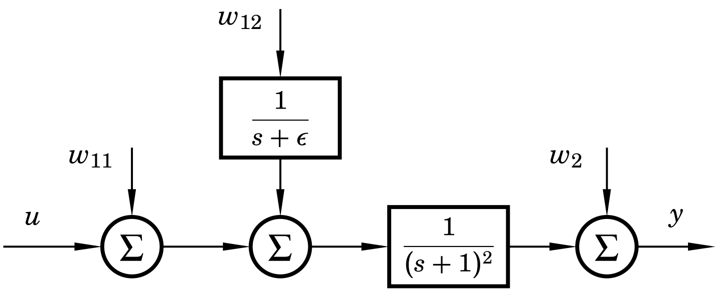

Figure 9.1 Process with additive low-frequency input disturbance.

controller to have small gain around the this frequency. This can be achieved by extending the measurement equation as

$$y_e(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t) + v_2(t) + v_3(t)$$

The extra measurement disturbance v_3 is generated by passing unit intensity white noise n through a second-order filter with a transfer function

$$H(s) = \frac{K_v s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

with $\omega_0 = 0.5$ rad/s. The parameter ζ determines the magnitude of the filter resonance peak, and we can choose e.g. $\zeta = 0.02$. Derive the extended state-space model

$$\begin{aligned} \dot{x}_e(t) &= A_e x_e(t) + B_e u(t) + N_e \begin{pmatrix} v_1(t) \\ n(t) \end{pmatrix} \\ y_e(t) &= C_e x_e(t) + v_2(t) \end{aligned}$$

and the associated noise intensity matrices needed to compute the Kalman filter.

b. **(*)** Compute the Kalman filter using `kalman` in Matlab. Plot the transfer function of the Kalman filter from y to $\hat{x}_1$ $(= \hat{y})$. Can you see the implication of the noise modeling?

9.6 (*) Consider the task of estimating the states of a double integrator where noise with intensity 1 affects the input only and we have measurement noise of intensity 1.

a. Determine the optimal Kalman filter.

b. What are the Kalman filter poles?

Solutions to Exercise 9. Kalman Filtering

9.1 **a.** With the problem parameters $A = C = 1$, the Riccati equation reduces to

$$2P + R_1 - \frac{P^2}{R_2} = 0,$$

which has the positive solution $P = R_2 + R_2\sqrt{1 + \frac{R1}{R2}}$. Thus, the Kalman filter gain is

$$K = \frac{1}{R_2}P = 1 + \sqrt{1 + \frac{R_1}{R_2}} = 1 + \sqrt{1+\beta}.$$

b. The Kalman filter error dynamics are given by

$$\begin{aligned}\dot{\tilde{x}}(t) &= (A - KC)\tilde{x}(t) + w_1(t) - Kw_2(t)\\ &= -\sqrt{1+\beta}\,\tilde{x}(t) + w_1(t) - (1 + \sqrt{1+\beta})w_2(t)\end{aligned}$$

c. The position of the Kalman filter pole is $-\sqrt{1+\beta}$. We can see that if $\beta \to \infty$, the pole of the Kalman filter $\to -\infty$. Hence, the estimation error dynamics are fast, and the Kalman filter very much trusts the measurements. On the other hand, if $\beta \to 0$, the Kalman filter pole tends to -1, that is, as fast as the process pole. Now, the filter trusts the process model much more than the measurements.

9.2 **a.** With the problem parameters

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad R_1 = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \quad R_2 = 1, \quad P = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}$$

the Riccati equation $AP + PA^T + R_1 - PC^TR_2^{-1}CP = 0$ leads to

$$\begin{aligned}-p_1^2 + 2p_2 + 3 &= 0\\ p_1 + p_3 - p_1p_2 &= 0\\ -p_2^2 + 2p_2 + 3 &= 0\end{aligned}$$

with the positive solution $p_1 = p_2 = 3, p_3 = 6$. The optimal P and K are thus

$$P = \begin{pmatrix} 3 & 3 \\ 3 & 6 \end{pmatrix}, \quad K = PC^T = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

b. The Kalman filter is given by $\frac{d\hat{x}}{dt} = (A - KC)\hat{x} + Bu + Ky$. Inserting the problem data and the optimal K gives

$$\begin{aligned}\frac{d\hat{x}_1}{dt} &= -3\hat{x}_1 + \hat{x}_2 + u + 3y\\ \frac{d\hat{x}_2}{dt} &= -2\hat{x}_1 + 3y\end{aligned}$$

c. See Matlab code below.

```
>> A = [0 1;1 0];
>> B = [1; 0];
>> C = [1 0];
>> R1 = [3 0; 0 3];
>> R2 = 1;
>> [K,P] = lqe(A,eye(2),C,R1,R2)
K =
    3.0000
    3.0000
P =
    3.0000    3.0000
    3.0000    6.0000
```

9.3 In each case, we are looking for the observer error covariance $\mathrm{E}\,\tilde{x}^2 = P$, where P is given by the solution to the algebraic Riccati equation

$$AP + PA^T + R_1 - (PC^T + R_{12})R_2^{-1}(PC^T + R_{12})^T = 0$$

a. In this case we have $A = 0$, $C = 1$, $R_1 = R_2 = 1$, $R_{12} = 0$ and the Riccati equation becomes

$$1 - P^2 = 0$$

with the solution $P = 1$.

b. In this case we have $A = 0$, $C = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$, $R_1 = 1$, $R_2 = \left(\begin{smallmatrix}1 & 0\\0 & 10\end{smallmatrix}\right)$, $R_{12} = \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$ and the Riccati equation becomes

$$1 - \tfrac{11}{10}P^2 = 0$$

with the solution $P = \sqrt{\frac{10}{11}} \approx 0.95$. Note that, by adding an independent sensor with large measurement noise, we can still reduce the observer error.

c. In this case we have $A = 0$, $C = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$, $R_1 = 1$, $R_2 = \left(\begin{smallmatrix}1 & 1\\1 & 10\end{smallmatrix}\right)$, $R_{12} = \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$ and the Riccati equation becomes

$$1 - P^2 = 0$$

with the solution $P = 1$. In this case, adding a second sensor does not help in reducing the observer error. An interpretation of the matrix R_2 is that the second sensor measures the same signal as the first sensor, plus some additional noise. Hence, the second signal contains no additional information.

9.4 Taking for instance x_1 as the output of $\frac{1}{s+\epsilon}$ and using x_2 and x_3 to realize $\frac{1}{(s+1)^2}$ we obtain

$$\begin{aligned}
\dot{x}_1 &= -\epsilon x_1 + w_{12} \\
\dot{x}_2 &= x_1 - x_2 + u + w_{11} \\
\dot{x}_3 &= -x_3 + x_2 \\
y &= x_3 + w_2
\end{aligned}$$

The relevant matrices for stating the Kalman filter Riccati equation are

$$A = \begin{pmatrix} -\epsilon & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}, \quad R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad R_2 = 1$$

9.5 **a.** We can choose for instance the controllable canonical form to realize the filter $H(s)$:

$$\dot{x}_H(t) = \begin{pmatrix} 0 & 1 \\ -0.25 & -0.02 \end{pmatrix} x_H(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} n(t)$$
$$v_3(t) = \begin{pmatrix} 0 & K_v \end{pmatrix} x_H(t)$$

Introducing the extended state vector $x_e = \begin{pmatrix} x \\ x_H \end{pmatrix}$ we can write the extended system as

$$\dot{x}_e(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -0.25 & -0.02 \end{pmatrix} x_e(t) + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 0.1 & 0 \\ 0.1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1(t) \\ n(t) \end{pmatrix}$$
$$y_e(t) = \begin{pmatrix} 1 & 0 & 0 & K_v \end{pmatrix} x_e(t) + v_2(t)$$

Thus, we have the intensity matrices $R_1 = \text{diag}(1,\ 1)$, $R_2 = 0.1$.

b. See Figure 9.1 for the Bode plot of the Kalman filter transfer function from measurement $y(t)$ to estimated process output $\hat{x}_1(t)$ using $K_v = 1$. We see a large attenuation of frequencies at $\omega = 0.5$ rad/s. Matlab code:

```
% Extended process model
A = [0 1 0 0; 0 -1 0 0; 0 0  0 1; 0 0 -0.25 -0.02];
B = [0; 1; 0; 0];
Kv = 1;
C = [1 0 0 Kv];
N = [0.1 0; 0.1 0; 0 0; 0 1]; % noise input matrix
R1 = eye(2);
R2 = 0.1;

% Design Kalman filter
sysk = ss(A,[B N],C,0);
kest = kalman(sysk,R1,R2);
Gx1hat_y = kest(2,2); % transfer function from y to x1hat
figure(1)
bode(Gx1hat_y)
```

9.6 **a.** One possible state-space realization is

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v_1(t)$$
$$y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} + v_2(t)$$

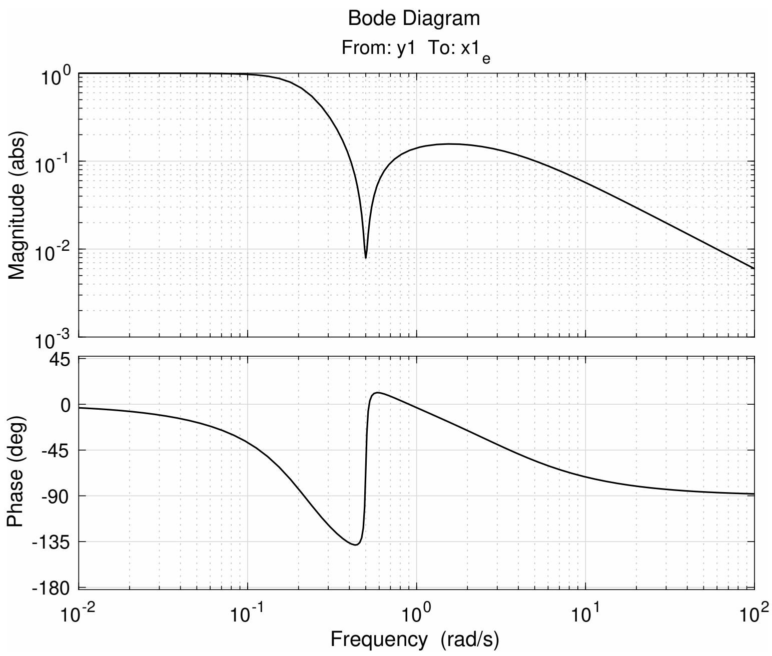

Figure 9.1 Kalman filter Bode diagram in Problem 9.5 b.

The Riccati equation

$$AP + PA^T + R_1 - PC^T R_2^{-1} CP = 0$$

is solved by letting $P = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}$. The equations become

$$\begin{aligned} 2p_2 - p_1^2 &= 0 \\ p_3 - p_1 p_2 &= 0 \\ 1 - p_2^2 &= 0 \end{aligned}$$

The positive solution is

$$P = \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}$$

with the optimal gain

$$K = PC^T = (\sqrt{2} \quad 1)^T$$

b. The poles of the Kalman filter are the eigenvalues of $A - KC$,

$$A - KC = \begin{pmatrix} -\sqrt{2} & 1 \\ -1 & 0 \end{pmatrix}$$

with the eigenvalues $\lambda_j = \dfrac{1}{\sqrt{2}}(-1 \pm i)$.