

## FRTN10 Exercise 8. Linear-Quadratic Control

8.1 Consider the first-order unstable process

$$\dot{x}(t) = ax(t) + u(t), \quad a > 0$$

a. Design an LQ controller  $u(t) = -Lx(t)$  that minimizes the criterion

$$J = \int_0^{\infty} (x^2(t) + \rho u^2(t)) dt, \quad \rho > 0$$

b. Calculate the location of the closed-loop as a function of  $\rho$  and discuss what happens when either  $\rho \rightarrow 0$  or  $\rho \rightarrow \infty$ .

8.2 Consider the second-order system

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} u(t) \\ y(t) &= (5 \ 5) x(t) \end{aligned}$$

Design an LQ controller  $u(t) = -Lx(t)$  that minimizes the criterion

$$J = \int_0^{\infty} (y^2(t) + 5u_1^2(t) + 5u_2^2(t)) dt.$$

What are the poles of the closed-loop system? Also calculate the minimal value of  $J$  when the initial state is  $x(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ .

8.3 Consider a process

$$\dot{x}(t) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 3 \\ 2 \end{pmatrix} u(t)$$

Show that  $u(t) = -Lx(t)$  with

$$L = (2 \ -3)$$

can *not* be an optimal state feedback designed using linear-quadratic control theory with the cost function

$$J = \int_0^{\infty} (x^T(t)Q_1x(t) + Q_2u^2(t)) dt$$

where  $Q_1, Q_2 > 0$ .

*Hint:* Sketch the Nyquist plot of the loop transfer function  $L(sI - A)^{-1}B$ .

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8.4 Consider the system

$$\dot{x} = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} x + \begin{pmatrix} -4 \\ 8 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} x$$

One wishes to minimize the criterion

$$J(T) = \int_0^T \left( x^T(t) Q_1 x(t) + Q_2 u^2(t) \right) dt$$

Is it possible to find positive definite weights  $Q_1$  and  $Q_2$  such that the cost function  $J(T) < \infty$  as  $T \rightarrow \infty$ ?

8.5 Consider the double integrator

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t)$$

A set of LQ controllers  $u(t) = -Lx(t) + L_r r(t)$  have been designed.  $L$  was calculated to minimize the cost function

$$J = \int_0^\infty \left( x^T(t) Q_1 x(t) + Q_2 u^2(t) \right) dt$$

and  $L_r$  was chosen to give unit static gain from  $r$  to  $y$ . The four plots in Figure 8.1 show the step responses of the closed-loop system for four different combinations of weights,  $Q_1, Q_2$ . Pair the combinations of weights given below with the step responses in Figure 8.1.

1)

$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_2 = 0.01$$

2)

$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_2 = 1$$

3)

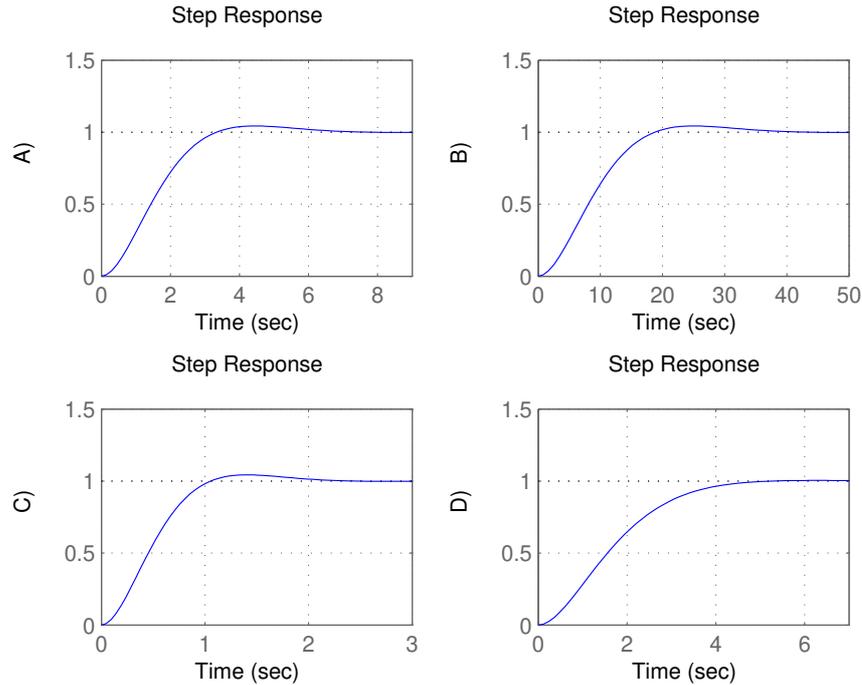
$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q_2 = 1$$

4)

$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_2 = 1000$$

8.6 (\*) We would like to control the following process with linear-quadratic optimal control:

$$\dot{x}(t) = \begin{pmatrix} 1 & 3 \\ 4 & 8 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0.1 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 0 & 1 \end{pmatrix} x(t)$$



**Figure 8.1** Step responses for LQ-control of the system in Problem 8.5 with different weights on  $Q_1, Q_2$ .

The penalty on  $x_1^2(t)$  should be 1, and the penalty on  $x_2^2(t)$  should be 2. For  $u^2(t)$  we will try different penalty values:  $\rho = 0.01, 1, 100$ .

- a. Determine the cost function for the three different cases.
- b.  Assume that we want to add reference tracking so that  $y = r$  in stationarity, using the control law  $u(t) = L_r r(t) - Lx(t)$ . In Matlab, calculate the three different resulting controllers, calculate the resulting closed-loop poles and simulate step responses from  $r$  to  $x_2$  and from  $r$  to  $u$ . Verify that there is no static error.

**8.7 (\*)** Consider the double integrator

$$\ddot{\xi}(t) = u(t).$$

with state-space representation

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x$$

where  $x = (\xi(t), \dot{\xi}(t))$ . You would like to design a controller using the criterion

$$\int_0^{\infty} (\xi^2(t) + \eta \cdot u^2(t)) dt$$

for some  $\eta > 0$ .

*Exercise 8. Linear-Quadratic Control*

**a.** Show that  $S = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}$  with

$$s_1 = \sqrt{2} \cdot \eta^{1/4}$$

$$s_2 = \eta^{1/2}$$

$$s_3 = \sqrt{2} \cdot \eta^{3/4}$$

solves the Riccati equation.

**b.** What are the closed-loop poles of the system when using this optimal state feedback? What happens with the control signal if  $\eta$  is reduced?

## Solutions to Exercise 8. Linear-Quadratic Control

8.1 a. Using  $A = a$ ,  $B = 1$ ,  $Q_1 = 1$ ,  $Q_2 = \rho$ ,  $Q_{12} = 0$  the Riccati equation becomes

$$2Sa + 1 - S\rho^{-1}S = 0$$

The positive solution is

$$S = a\rho + \sqrt{(a\rho)^2 + \rho}$$

and the optimal controller gain is given by

$$L = \frac{S}{\rho} = a + \sqrt{a^2 + \frac{1}{\rho}}.$$

b. The closed-loop pole is given by

$$A - BL = -\sqrt{a^2 + \frac{1}{\rho}}$$

The pole is located in the left half-plane for all  $\rho > 0$ . When  $\rho \rightarrow 0$ , control is cheap and the gain approaches  $\infty$ , while the pole approaches  $-\infty$ . When  $\rho \rightarrow \infty$ , control is expensive and the gain approaches 2, while the pole approaches  $-a$  (actually the open-loop pole mirrored in the imaginary axis). It is interesting to note that, since the system is unstable, the gain cannot approach zero when control becomes expensive.

8.2 Using  $y(t) = Cx(t)$  and  $u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$  we first rewrite the cost function as

$$J = \int_0^{\infty} \left( x^T(t)C^T Cx(t) + u^T(t) \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} u(t) \right) dt$$

from which we identify  $Q_1 = C^T C = \begin{pmatrix} 25 & 25 \\ 25 & 25 \end{pmatrix}$ ,  $Q_2 = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$  and  $Q_{12} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ . The Riccati equation is  $Q_1 + A^T S + SA - SBQ_2^{-1}B^T S = 0$ . Let

$$S = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}$$

We get the following system of equations:

$$\begin{aligned} -s_1^2 + 2s_1 + 2s_2 + 25 &= 0 \\ s_2 + s_3 - s_1s_2 + 25 &= 0 \\ 25 - s_2^2 &= 0 \end{aligned}$$

The positive solution is  $s_1 = 7$ ,  $s_2 = s_3 = 5$ . This gives the state feedback gain

$$L = Q_2^{-1}B^T S = \begin{pmatrix} 2.8 & 2 \\ 1.4 & 1 \end{pmatrix}.$$

The poles of the closed-loop system are given by  $\det(\lambda I - A + BL) = 0$  which gives  $\lambda_1 = -1$ ,  $\lambda_2 = -5$ .

The minimal value of  $J$  is given by  $x^T(0)Sx(0) = 2$ .

**8.3** The loop gain is

$$L(sI - A)^{-1}B = \frac{6}{(s+1)(s+2)}$$

The Nyquist curve starts on the positive real axis and will approach the origin along the negative real axis with phase  $-180^\circ$  as  $\omega \rightarrow \infty$ . This is not consistent with an LQ-optimal loop gain, which will always remain at a distance  $\geq 1$  from the critical point  $-1$  and will hence have an asymptotic phase of  $-90^\circ$ . Therefore,  $L$  cannot be an LQ-optimal state feedback vector.

**8.4** The system has two unstable poles in 2 and 3. If the cost function should be less than  $\infty$  then the system must be stabilizable, i.e. all unstable poles must be controllable (due to  $Q_1 > 0$ ). The controllability matrix is given by

$$W_c = (B \ AB) = \begin{pmatrix} -4 & -12 \\ 8 & 24 \end{pmatrix}$$

which is a rank 1 matrix. Thus, only one of the modes is controllable meaning that there is an uncontrollable, unstable mode, and hence, we can not make the cost function less than  $\infty$ .

**8.5** 3) is the only case with a cost on the velocity  $x_2$ . This makes the controller try to avoid rapid variations in  $x_1$ , so we get 3)  $- D$ ), the only step response without overshoot. The weight,  $Q_2$ , on the control signal determines the speed of the system. A low weight on the control signal gives a faster system since we are allowed to use more control signal. This reveals 1)  $- C$ ), 2)  $- A$ ), 4)  $- B$ ).

**8.6 a.** The cost function is

$$J = \int_0^\infty \left( x^T(t) \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} x(t) + \rho u^2(t) \right) dt, \quad \rho = 0.01, 10, 1000$$

**b.** See Figure 8.1 for step responses, and Matlab code below.

```
A = [1 3; 4 8]; B = [1; 0.1]; C = [0 1];
P = ss(A,B,C,0);

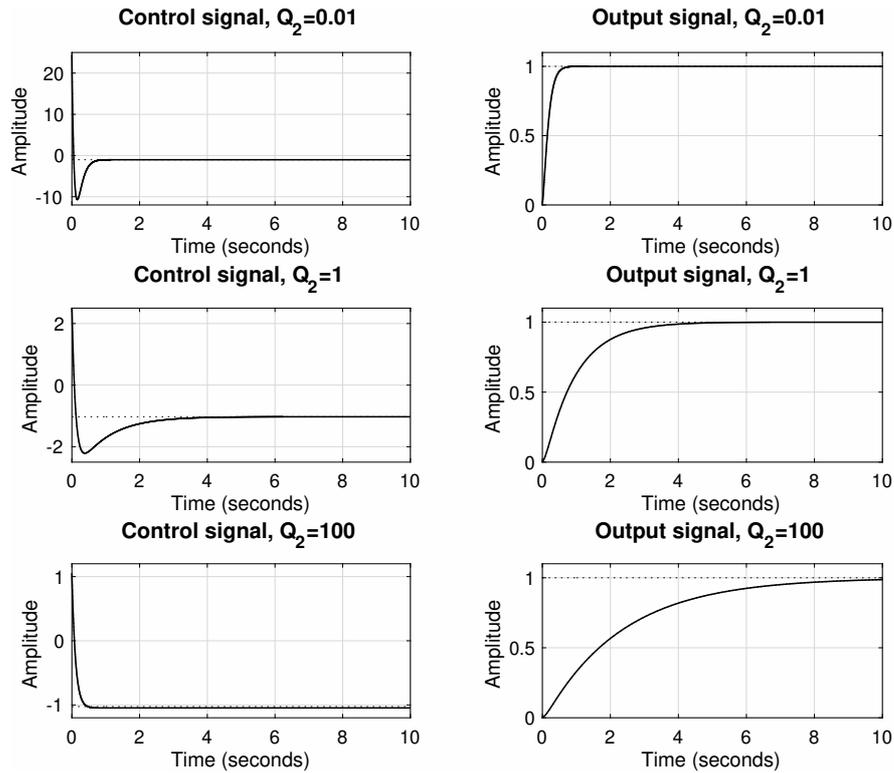
Q1 = [1 0; 0 2]; Q2_vector = [0.01 1 100];

clf
for i=1:length(Q2_vector)
[L,S,E] = lqr(P,Q1,Q2_vector(i));

% Calculating Lr (static gain to output should be 1)
Lr = 1/(C/(B*L-A)*B);

% Closed loop from r to u:
Gur = ss(A-B*L,B*Lr,-L,Lr);

% Closed loop from r to y:
```



**Figure 8.1** Step responses for different weight on control signal.

```
Gyr = ss(A-B*L,B*Lr,C,0);
```

```
% Plotting step responses
```

```
subplot(3,2,i*2-1)
```

```
step(Gur)
```

```
axis([0 10 -Inf Inf])
```

```
title(['Control signal, Q_2=' num2str(Q2_vector(i))])
```

```
subplot(3,2,i*2)
```

```
step(Gyr)
```

```
axis([0 10 -Inf Inf])
```

```
title(['Output signal, Q_2=' num2str(Q2_vector(i))])
```

```
poles{i} = E;
```

```
end
```

```
poles{:}
```

- 8.7 a.** Weighting matrices  $Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  and  $Q_2 = \eta$ . The Riccati equation to be solved with respect to  $S$  is

$$A^T S + SA + Q_1 - SBQ_2^{-1}B^T S = 0$$

Put

$$S = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix},$$

which gives

$$\begin{pmatrix} 0 & 0 \\ s_1 & s_2 \end{pmatrix} + \begin{pmatrix} 0 & s_1 \\ 0 & s_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \frac{1}{\eta} \cdot \begin{pmatrix} s_2^2 & s_2 s_3 \\ s_2 s_3 & s_3^2 \end{pmatrix} = 0$$

We see, by insertion, that

$$\begin{aligned} s_1 &= \sqrt{2} \cdot \eta^{1/4} \\ s_2 &= \eta^{1/2} \\ s_3 &= \sqrt{2} \cdot \eta^{3/4} \end{aligned}$$

solves the Riccati equation.

**b.** The optimal state feedback is

$$\begin{aligned} L &= Q_2^{-1} B^T S = \frac{1}{\eta} \cdot (0 \quad 1) \begin{pmatrix} \sqrt{2}\eta^{1/4} & \eta^{1/2} \\ \eta^{1/2} & \sqrt{2} \cdot \eta^{-3/4} \end{pmatrix} \\ &= \frac{1}{\eta} \cdot (\eta^{1/2} \quad \sqrt{2}\eta^{3/4}) = (\eta^{-1/2} \quad \sqrt{2} \cdot \eta^{-1/4}) \end{aligned}$$

The poles are the eigenvalues to  $A - BL$ . Put  $\mu = \eta^{-1/4} \Rightarrow L = (\mu^2 \quad \sqrt{2} \cdot \mu)$ . This gives

$$0 = \det \begin{pmatrix} s & -1 \\ \mu^2 & s + \sqrt{2} \cdot \mu \end{pmatrix} = s^2 + \sqrt{2}\mu s + \mu^2,$$

that is

$$\begin{aligned} s &= -\frac{\mu}{\sqrt{2}} \pm \sqrt{\frac{\mu^2}{2} - \mu^2} = -\frac{\mu}{\sqrt{2}} \pm i \cdot \frac{\mu}{\sqrt{2}} = \\ &= -\frac{\mu}{\sqrt{2}} \cdot (1 \pm i) = -\frac{1}{\sqrt{2} \cdot \eta^{1/4}} \cdot (1 \pm i) \end{aligned}$$

If  $\eta$  is reduced, the distance between the poles and the origin will increase. This means that the size of  $u(t)$  will increase.