FRTN10 Exercise 7. Controller Structures, Preparations for Lab 2

Note: Exercises 7.1–7.3 serve as preparation for Laboratory Excercise 2.

You may choose to use Matlab to solve any of the problems or just to verify your calculations.

7.1 **a.** Give the definition of RGA for a complex-valued, not necessarily square, matrix A. How do you apply it to a process $G(s)$ and what information can be extracted in an automatic control perspective?

b. Let

$$G(s) = \begin{pmatrix} \dfrac{1}{s+2} & \dfrac{10}{s+1} \\ \dfrac{1}{s+5} & \dfrac{5}{s+3} \end{pmatrix}.$$

Compute RGA($G(0)$). What input-output pairing would you recommend in a decentralised control structure?

7.2 Consider the MIMO process

$$P(s) = \begin{pmatrix} \dfrac{1}{s+1} & 0 & 0 \\ 0 & \dfrac{0.1}{s+10} & \dfrac{1}{s+10} \\ \dfrac{0.1}{s+1} & \dfrac{1}{s+1} & 0 \end{pmatrix}.$$

Compute the relative gain array, RGA, of $P(0)$ and suggest an input-output pairing for the system based on this.

Hint: The inverse of $P(s)$ is given by

$$P(s)^{-1} = \begin{pmatrix} s+1 & 0 & 0 \\ -0.1(s+1) & 0 & s+1 \\ 0.01(s+1) & s+10 & -0.1(s+1) \end{pmatrix}.$$

7.3 Figure 7.1 shows the quadruple-tank process that will be used in Lab 2. The goal is to control the measured levels in the lower tanks (y_1, y_2) using the pumps (u_1, u_2). For each tank $i = 1 \dots 4$, mass balance and Torricelli's law give that

$$A_i \frac{dh_i}{dt} = -a_i\sqrt{2gh_i} + q_{in} \tag{7.1}$$

where A_i is the cross-section of the tank, h_i is the water level, a_i is the cross-section of the outlet hole, g is the acceleration of gravity, and q_{in} is the inflow to the tank. The non-linear equation (7.1) can be linearized around a stationary point (h_i^0, q_{in}^0), giving the linear equation

$$A_i \frac{d\Delta h_i}{dt} = -a_i\sqrt{\frac{g}{2h_i^0}}\,\Delta h_i + \Delta q_{in} \tag{7.2}$$

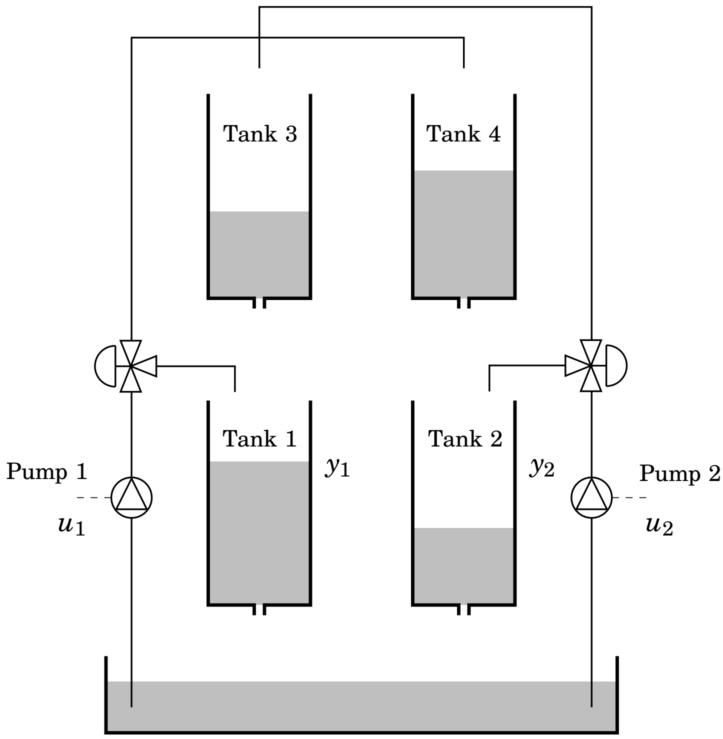

Figure 7.1 The quadruple-tank process.

where $\Delta h_i = h_i - h_i^0$, and $\Delta q_{in} = q_{in} - q_{in}^0$ denote deviations around the stationary point.

The flows from the pumps are divided according to two parameters $\gamma_1, \gamma_2 \in (0, 1)$. The flow to Tank 1 is $\gamma_1 k_1 u_1$ and the flow to Tank 4 is $(1-\gamma_1)k_1u_1$. Symmetrically, the flow to Tank 2 is $\gamma_2 k_2 u_2$ and the flow to Tank 3 is $(1-\gamma_2)k_2u_2$.

a. Let $\Delta u_i = u_i - u_i^0$, $\Delta h_i = h_i - h_i^0$, and $\Delta y_i = y_i - y_i^0$. Verify that the linearized dynamics of the complete quadruple-tank system are given by

$$
\begin{aligned}
\frac{d\Delta h_1}{dt} &= -\frac{a_1}{A_1}\sqrt{\frac{g}{2h_1^0}}\,\Delta h_1 + \frac{a_3}{A_1}\sqrt{\frac{g}{2h_3^0}}\,\Delta h_3 + \frac{\gamma_1 k_1}{A_1}\,\Delta u_1 \\
\frac{d\Delta h_2}{dt} &= -\frac{a_2}{A_2}\sqrt{\frac{g}{2h_2^0}}\,\Delta h_2 + \frac{a_4}{A_2}\sqrt{\frac{g}{2h_4^0}}\,\Delta h_4 + \frac{\gamma_2 k_2}{A_2}\,\Delta u_2 \\
\frac{d\Delta h_3}{dt} &= -\frac{a_3}{A_3}\sqrt{\frac{g}{2h_3^0}}\,\Delta h_3 + \frac{(1-\gamma_2)k_2}{A_3}\,\Delta u_2 \\
\frac{d\Delta h_4}{dt} &= -\frac{a_4}{A_4}\sqrt{\frac{g}{2h_4^0}}\,\Delta h_4 + \frac{(1-\gamma_1)k_1}{A_4}\,\Delta u_1
\end{aligned}
$$

Introduce the input vector, u, output vector, y, and state vector, x, as

$$
u = \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix}, \quad x = \begin{pmatrix} \Delta h_1 \\ \Delta h_2 \\ \Delta h_3 \\ \Delta h_4 \end{pmatrix}, \quad y = \begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix}.
$$

Verify that the linearized system can be written in state-space form as

$$\frac{dx}{dt} = \begin{pmatrix} -\frac{1}{T_1} & 0 & \frac{A_3}{A_1 T_3} & 0 \\ 0 & -\frac{1}{T_2} & 0 & \frac{A_4}{A_2 T_4} \\ 0 & 0 & -\frac{1}{T_3} & 0 \\ 0 & 0 & 0 & -\frac{1}{T_4} \end{pmatrix} x + \begin{pmatrix} \frac{\gamma_1 k_1}{A_1} & 0 \\ 0 & \frac{\gamma_2 k_2}{A_2} \\ 0 & \frac{(1-\gamma_2)k_2}{A_3} \\ \frac{(1-\gamma_1)k_1}{A_4} & 0 \end{pmatrix} u,$$

$$y = \begin{pmatrix} k_c & 0 & 0 & 0 \\ 0 & k_c & 0 & 0 \end{pmatrix} x,$$

where $T_i = \frac{A_i}{a_i}\sqrt{\frac{2h_i^0}{g}}$, and k_c is a measurement constant.

b. Show that the transfer matrix from u to y is given by

$$P(s) = \begin{pmatrix} \frac{\gamma_1 c_1}{1+sT_1} & \frac{k_2}{k_1}\cdot\frac{(1-\gamma_2)c_1}{(1+sT_1)(1+sT_3)} \\ \frac{k_1}{k_2}\cdot\frac{(1-\gamma_1)c_2}{(1+sT_2)(1+sT_4)} & \frac{\gamma_2 c_2}{1+sT_2} \end{pmatrix}$$

where $c_1 = T_1 k_1 k_c / A_1$ and $c_2 = T_2 k_2 k_c / A_2$.

Hint: Use the fact that

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & c & 0 & d \\ 0 & 0 & e & 0 \\ 0 & 0 & 0 & f \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a} & 0 & -\frac{b}{ae} & 0 \\ 0 & \frac{1}{c} & 0 & -\frac{d}{cf} \\ 0 & 0 & \frac{1}{e} & 0 \\ 0 & 0 & 0 & \frac{1}{f} \end{pmatrix}$$

c. The zeros are given by the equation

$$\det P(s) = \frac{c_1 c_2 \big(\gamma_1\gamma_2(1+sT_3)(1+sT_4) - (1-\gamma_1)(1-\gamma_2)\big)}{(1+sT_1)(1+sT_2)(1+sT_3)(1+sT_4)} = 0$$

which is simplified to

$$(1+sT_3)(1+sT_4) - \frac{(1-\gamma_1)(1-\gamma_2)}{\gamma_1\gamma_2} = 0.$$

Show that the system is minimum phase (i.e., that both zeros are stable) if $1 < \gamma_1 + \gamma_2 < 2$, and that the system is non-minimum phase (i.e., that at least one zero is unstable) if $0 < \gamma_1 + \gamma_2 < 1$. Remember that $\gamma_1, \gamma_2 \geq 0$.

Hint: A second-order polynomial has all of its roots in the left half plane if and only if all coefficients have the same sign.

In the lab, we will first study the case $\gamma_1 = \gamma_2 \approx 0.7$, and then the case $\gamma_1 = \gamma_2 \approx 0.3$. In which case will the process be more difficult to control?

d. Show that the RGA for $P(0)$ is given by

$$\begin{pmatrix} \lambda & 1-\lambda \\ 1-\lambda & \lambda \end{pmatrix}$$

where $\lambda = \gamma_1\gamma_2/(\gamma_1+\gamma_2-1)$.
Based on this RGA matrix, suggest an input-output pairing in the two cases $\gamma_1 = \gamma_2 \approx 0.7$ and $\gamma_1 = \gamma_2 \approx 0.3$.

7.4 Consider the following multivariable system

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \dfrac{1}{10s+1} & \dfrac{-2}{2s+1} \\ \dfrac{1}{10s+1} & \dfrac{s-1}{2s+1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

a. By using RGA at stationarity, decide the input-output pairing that should be used in a decentralized control structure.

b. Assume that we want to use decentralized control, that is, we want to use a controller that can be described by

$$F^{\text{diag}}(s) = \begin{pmatrix} F_{11}(s) & 0 \\ 0 & F_{22}(s) \end{pmatrix}.$$

Also, we want the control loops to be decoupled in stationarity. Give the structure of such a controller $F(s)$ expressed in $F^{\text{diag}}(s)$ that is capable to do so. *Hint: Use a suitable decoupling matrix.*

7.5 (*) In this exercise we will try to design controllers for a 2×2-process, that is, a process that has 2 inputs and 2 outputs. The process is described by the transfer function matrix

$$G(s) = \begin{pmatrix} \dfrac{4}{s+1} & \dfrac{3}{3s+1} \\ \dfrac{1}{3s+1} & \dfrac{2}{s+0.5} \end{pmatrix}.$$

Design two different decentralized controllers for the process.

1. Decentralized control, using the RGA of the process.
2. Decentralized control, using decoupling with respect to stationarity

In both cases, use ordinary PI controllers. Use the step responses to evaluate the performance of the loop.

Solutions to Exercise 7. Controller Structures, Preparations for Lab 2

7.1 a. The relative gain array for a general, complex-valued matrix A is given by

$$\mathrm{RGA}\,(A) = A\,.\!*(A^{\dagger})^T$$

where † denotes the pseudo-inverse of A, and $.*$ denotes element-wise multiplication. For a process $G(s)$ the RGA is most often computed for the static gain $G(0)$ and sometimes also for the intended cross-over frequency, $G(i\omega_c)$. By inspecting the elements in the RGA-matrix, we get advice on what output should be controlled using what input. We should choose pairings that have relative gains close to 1 and avoid pairings that have negative relative gains.

b.

$$\mathrm{RGA}(G(0)) = G(0)\,.*\,G^{-T}(0) = \begin{pmatrix} -\frac{5}{7} & \frac{12}{7} \\ \frac{12}{7} & -\frac{5}{7} \end{pmatrix}$$

Since we should avoid negative relative gains we should choose the pairing $y_1 \leftrightarrow u_2$ and $y_2 \leftrightarrow u_1$.

7.2 We have

$$P(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.01 & 0.1 \\ 0.1 & 1 & 0 \end{pmatrix}$$

and

$$P(0)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -0.1 & 0 & 1 \\ 0.01 & 10 & -0.1 \end{pmatrix}$$

$$\mathrm{RGA}(P(0)) = P(0)\,.*\,(P(0)^{-1})^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

The RGA suggests that we should control output 1 with input 1, output 2 with input 3, and output 3 with input 2.

7.3 a. We see from the flow equation

$$A_i \frac{d\Delta h_i}{dt} = -a_i \sqrt{\frac{g}{2h_i^0}}\;\Delta h_i + \Delta q_{in} \tag{7.1}$$

that the outflow from tank i is

$$q_{out} = a_i \sqrt{\frac{g}{2h_i^0}}\;\Delta h_i.$$

The inflows into the tanks are found as the sum of the outflow from the tank above and the flow from the pumps into the respective tanks. Writing down equation (7.1) for each of the four tanks now gives the dynamics.

Substituting the time constants T_i into the dynamics, and arranging them into matrix form then gives the state-space form.

b. The transfer matrix is given by

$$P(s) = C(sI - A)^{-1}B =$$

$$= \begin{pmatrix} k_c & 0 & 0 & 0 \\ 0 & k_c & 0 & 0 \end{pmatrix} \begin{pmatrix} s + \frac{1}{T_1} & 0 & -\frac{A_3}{A_1 T_3} & 0 \\ 0 & s + \frac{1}{T_2} & 0 & -\frac{A_4}{A_2 T_4} \\ 0 & 0 & s + \frac{1}{T_3} & 0 \\ 0 & 0 & 0 & s + \frac{1}{T_4} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\gamma_1 k_1}{A_1} & 0 \\ 0 & \frac{\gamma_2 k_2}{A_2} \\ 0 & \frac{(1-\gamma_2)k_2}{A_3} \\ \frac{(1-\gamma_1)k_1}{A_4} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\gamma_1 c_1}{1+sT_1} & \frac{k_2}{k_1} \cdot \frac{(1-\gamma_2)c_1}{(1+sT_1)(1+sT_3)} \\ \frac{k_1}{k_2} \cdot \frac{(1-\gamma_1)c_2}{(1+sT_2)(1+sT_4)} & \frac{\gamma_2 c_2}{1+sT_2} \end{pmatrix}$$

c. The zeros are given by the equation

$$T_3 T_4 s^2 + (T_3 + T_4)s + 1 - \frac{(1-\gamma_1)(1-\gamma_2)}{\gamma_1 \gamma_2} = 0$$

The two first coefficients are always positive, since $T_3, T_4 > 0$. The last coefficient is positive (and both zeros are thus stable) iff

$$\frac{(1-\gamma_1)(1-\gamma_2)}{\gamma_1 \gamma_2} < 1 \quad \Leftrightarrow \quad \gamma_1 + \gamma_2 > 1$$

In the case $\gamma_1 = \gamma_2 = 0.7$ we get a minimum-phase system which should be easier to control than the non-minimum-phase system we get in the case $\gamma_1 = \gamma_2 = 0.3$.

d. We have

$$P(0) = \begin{pmatrix} \gamma_1 c_1 & \frac{k_2}{k_1}(1-\gamma_2)c_1 \\ \frac{k_1}{k_2}(1-\gamma_1)c_2 & \gamma_2 c_2 \end{pmatrix}$$

and

$$P(0)^{-1} = \frac{1}{c_1 c_2(\gamma_1 + \gamma_2 - 1)} \begin{pmatrix} \gamma_2 c_2 & -\frac{k_2}{k_1}(1-\gamma_2)c_1 \\ -\frac{k_1}{k_2}(1-\gamma_1)c_2 & \gamma_1 c_1 \end{pmatrix}$$

$$\begin{aligned} \text{RGA}(P(0)) &= P(0) .* (P(0)^{-1})^T = \\ &= \frac{1}{c_1 c_2(\gamma_1+\gamma_2-1)} \begin{pmatrix} \gamma_1 c_1 \gamma_2 c_2 & -(1-\gamma_2)c_1(1-\gamma_1)c_2 \\ -(1-\gamma_2)c_1(1-\gamma_1)c_2 & \gamma_2 c_2 \gamma_1 c_1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2-1} & 1 - \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2-1} \\ 1 - \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2-1} & \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2-1} \end{pmatrix} = \begin{pmatrix} \lambda & 1-\lambda \\ 1-\lambda & \lambda \end{pmatrix} \end{aligned}$$

In the case $\gamma_1 = \gamma_2 = 0.7$ we get

$$\mathrm{RGA}(P(0)) = \begin{pmatrix} 1.225 & -0.225 \\ -0.225 & 1.225 \end{pmatrix}$$

The RGA suggests we should control output 1 with input 1 and output 2 with input 2.

In the case $\gamma_1 = \gamma_2 = 0.3$ we get

$$\mathrm{RGA}(P(0)) = \begin{pmatrix} -0.225 & 1.225 \\ 1.225 & -0.225 \end{pmatrix}$$

The RGA suggests that in this case we should control output 1 with input 2 and output 2 with input 1.

7.4 **a.** We compute the RGA for stationarity, i.e. $s = 0$.

$$\mathrm{RGA}(G(s)) = \begin{pmatrix} \frac{s-1}{s+1} & \frac{2}{s+1} \\ \frac{2}{s+1} & \frac{s-1}{s+1} \end{pmatrix}$$

gives

$$\mathrm{RGA}(G(0)) = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}.$$

Since you should avoid pairing that gives negative diagonal elements we choose $y_1 \leftrightarrow u_2$ and $y_2 \leftrightarrow u_1$.

b. We have that

$$G(0) = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$$

Using a decoupled controller structure with $W_1 = G^{-1}(0)$ and $W_2 = I$ we get a decoupled system in stationarity. The controller is

$$F(s) = W_1 F^{\mathrm{diag}}(s) W_2 = \begin{pmatrix} -F_{11}(s) & 2F_{22}(s) \\ -F_{11}(s) & F_{22}(s) \end{pmatrix}.$$

7.5 1. Decentralized control. First we calculate the RGA of the process,

$$\mathrm{RGA}(G(0)) = G(0) .* G^{-T}(0) = \begin{pmatrix} 1.2308 & -0.2308 \\ -0.2308 & 1.2308 \end{pmatrix}.$$

We see that we should choose $y_1 \leftrightarrow u_1$ and $y_2 \leftrightarrow u_2$. A resonable tuning, either by pole placement or hand tuning, gives PI controllers with parameters close to

$$F(s) = \begin{pmatrix} 2(1 + \frac{1}{0.5s}) & 0 \\ 0 & 2(1 + \frac{1}{0.5s}) \end{pmatrix}.$$

See figure 7.1 for step responses.

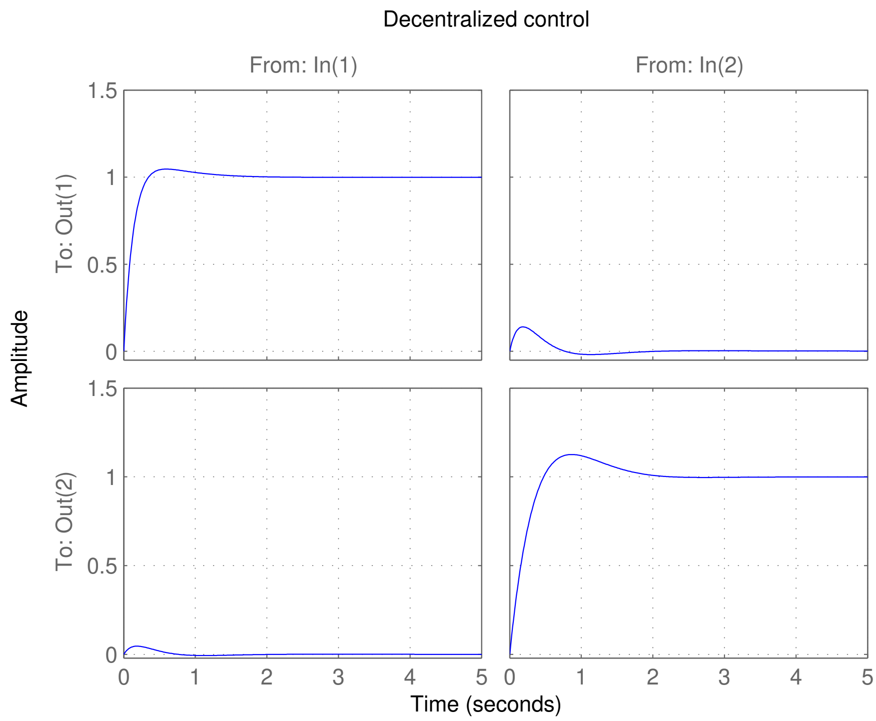

Figure 7.1 Decentralized control

2. Decoupled control. The inverse of the static gain matrix is given by

$$G^{-1}(0) = \begin{pmatrix} 4 & 3 \\ 1 & 4 \end{pmatrix}^{-1}$$

Thus, for decoupling, we use $W_1 = G^{-1}(0)$ and $W_2 = I$. Hand-tuning of the PI controllers gives

$$F(s) = \begin{pmatrix} 40(1 + \frac{1}{0.5s}) & 0 \\ 0 & 20(1 + \frac{1}{0.8s}) \end{pmatrix}.$$

See figure 7.2 for step responses.

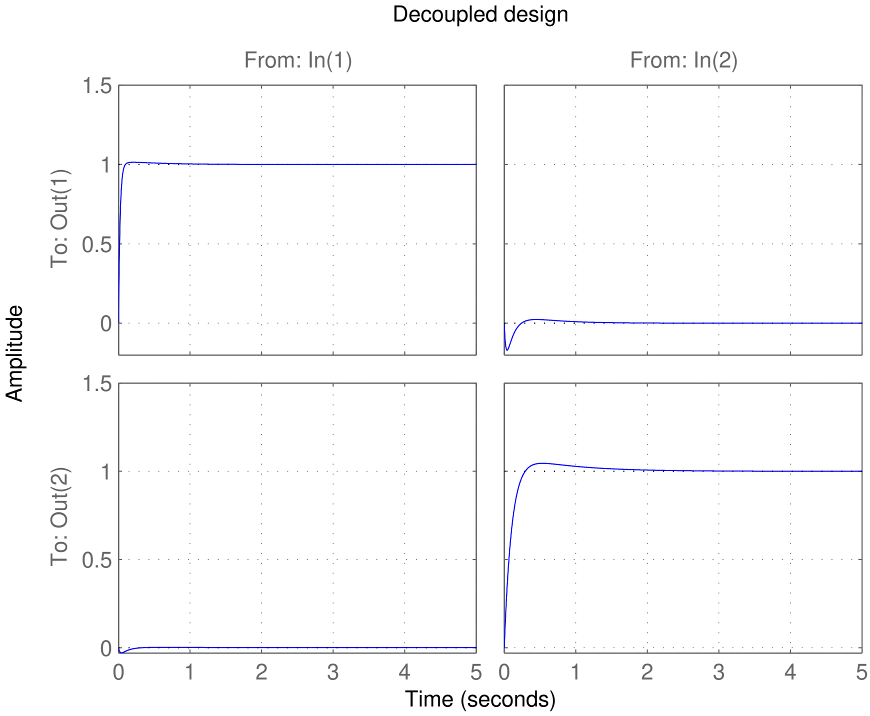

Figure 7.2 Decoupled control

Matlab code:

```
s = tf('s');
G = [4/(s+1) 3/(3*s+1); 1/(3*s+1) 2/(s+0.5)];

% Decentralized control
RGA = dcgain(G).*(inv(dcgain(G))).'
F = [2*(1+1/(0.5*s)) 0;0 2*(1+1/(0.5*s))];
figure(1)
step(feedback(G*F, eye(2)),5)
title('Decentralized control');grid

% Decoupled design
Go = dcgain(G)
F = [40*(1+1/(0.5*s)) 0;0 20*(1+1/(0.8*s))];
figure(2);
step(feedback(G*inv(Go)*F,eye(2)),5);
title('Decoupled design');grid
```