FRTN10 Exercise 6. Fundamental Limitations

6.1 Consider the ball in the hoop in Figure 6.1. This process consists of a cylinder rotating with the angular velocity ω. Inside the cylinder, a ball is rolling. The position of the ball is given by the angle θ and the linearized dynamics can be written as $\ddot{\theta} + c\dot{\theta} + k\theta = \dot{\omega}$. Let $k = 1$ and $c = 2$.

a. What is the transfer function from cylinder velocity ω to the position θ of the ball? Where is the zero located?

b. What limitation on the sensitivity function for an asymptotically stable closed-loop system is imposed by the process zero?

c. What consequence does the process zero have on the static error when a reference signal $r(t)$, e.g. a step with the magnitude a, is to be followed. Let the control signal $\omega(t)$ be determined from the error signal $r(t) - \theta(t)$ via the controller transfer function $C(s)$. Give a physical interpretation.

d. Usually an integrator is introduced in the controller in order to remove static errors. How would the ball/hoop-system behave with a PI controller and a non-zero reference position for the ball?

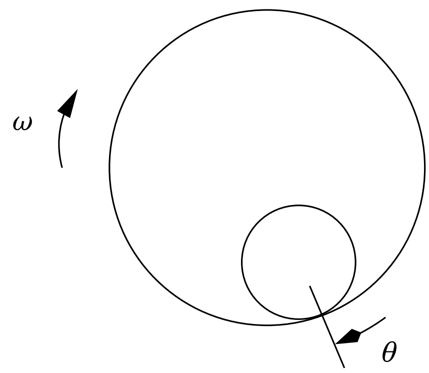

Figure 6.1 The ball in the hoop.

6.2 A resonant mechanical system has the pole-zero configuration shown in Figure 6.2. The controller structure is given by Figure 6.3.

a. What constraint does a purely imaginary process pole in $i\omega_p$ impose on the sensitivity function?

b. What consequence does this give for the control error e, in presence of a sinusoidal measurement disturbance n with frequency ω_p?

c. What effect does the controller $C(s)$ have on this response?

d. What constraint does a purely imaginary process zero in $i\omega_z$ impose on the sensitivity function?

e. What consequence does this give for the response z, in presence of a sinusoidal measurement disturbance n with a frequency ω_z?

6.3 Consider the setup in Figure 6.3 with $P(s) = (3 - s)/(s + 1)^2$

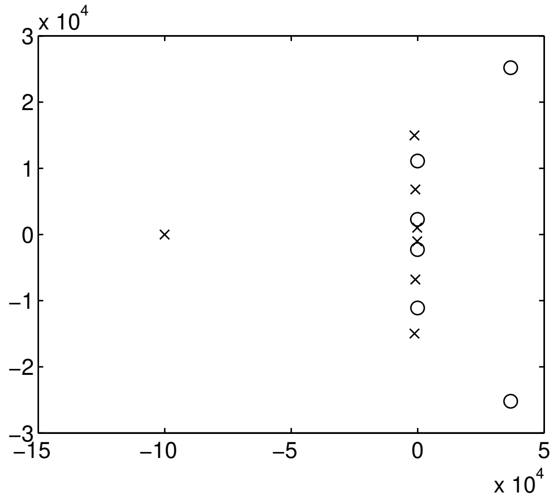

Figure 6.2 Pole-zero configuration of a resonant mechanical system.

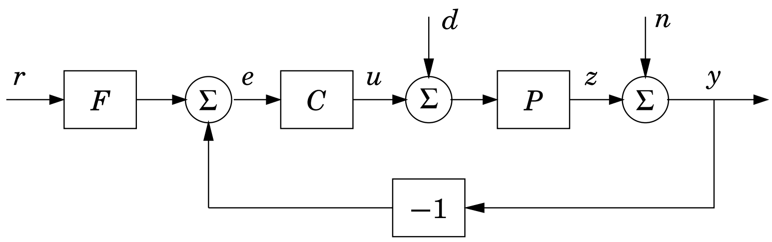

Figure 6.3 Two degree of freedom controller structure.

a. Does there exist a stabilizing controller $C(s)$ such that the transfer function from n to z becomes $5/(s+5)$? (Note: All transfer functions in the gang of four must be stable.)

b. Show that the specification

$$|S(i\omega)| \le \frac{2\omega}{\sqrt{\omega^2+36}} \qquad \omega \in \mathbb{R}^+$$

is equivalent to

$$\sup_\omega |W_a(i\omega)S(i\omega)| \le 1$$

with $a = 6$ and

$$W_a(s) = \frac{s+a}{2s}$$

Is this specification possible to satisfy ?

c. **(*)** Use Matlab to find a stabilizing controller $C(s)$ such that

$$\left|\frac{1}{1+P(i\omega)C(i\omega)}\right| \le \frac{2\omega}{\sqrt{\omega^2+1}} \qquad \text{for } \omega \in [0,2]$$

Hint: Use a PI controller of the form:

$$C(s) = K\frac{s/b+1}{s}$$

6.4 For each of the following three design problems, state if it is possible to construct a controller that can achieve the given specification. Motivate your answers! *Hint:* It is *not* possible in *at least* two of the cases.

System	Specification
$P_1(s) = \frac{e^{-2s}}{s+2}$	The step response must reach 0.9 before $t = 1$.
$P_2(s) = 3\frac{(s+40)(s-20)}{s^2(s-10)}$	The gain curve of the closed-loop transfer function T should lie between the two gain curves depicted in Figure 6.4.
$P_3(s) = \frac{1}{s-3}$	The step response must stay in the interval $[0, 2]$ for all t.

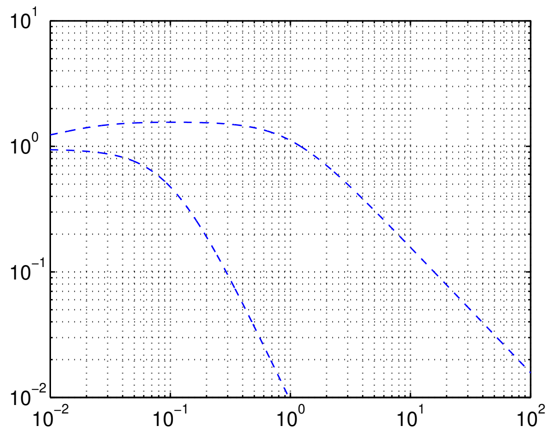

Figure 6.4 Gain specification for the closed-loop transfer function T in Problem 6.4.

6.5 **(*)** The specifications

$$\sup_\omega |W_S(i\omega)S(i\omega)| \le 1 \qquad \sup_\omega |W_T(i\omega)T(i\omega)| \le 1$$

where $S(s)$ and $T(s)$ are stable and minimum-phase transfer functions, can be used to make sure that the sensitivity is small in a low frequency range and measurement noise is rejected in a high frequency range.

a. Show that the two specifications are incompatible if

$$|W_S(s)| = |W_T(s)| > 2$$

for some right half plane s. (Hint: Use the Maximum Modulus Theorem and the fact that $S + T = 1$.)

b. Show that the two specifications are incompatible if

$$W_S(s) = \left(\frac{s+0.1}{s}\right)^n \qquad W_T(s) = \left(\frac{s+10}{10}\right)^n$$

and $n \ge 8$.

Hint: Use the result of **a**.

6.6

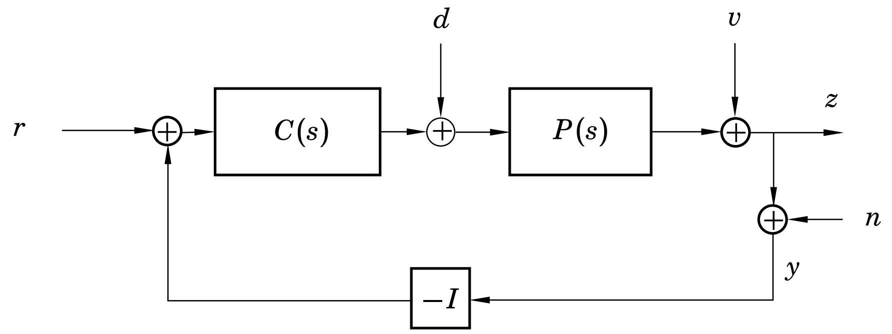

Figure 6.5 System in Problem 6.6

A multi-variable system (block diagram in Figure 6.5) is supposed to attenuate all output load disturbances (v) with at least a factor 10 for frequencies *below* 0.1 rad/sec. Constant output load disturbances should be attenuated by at least a factor 100 in stationarity.

Furthermore, the system should also attenuate measurement disturbances (n) with at least a factor 10 for frequencies *above* 2 rad/sec.

a. Formulate specifications on (the singular values of) S and T that guarantee the above requirements.

b. Re-formulate the specifications in **a** using $\|\cdot\|_\infty$ and the weighting functions W_S and W_T.

6.7 Consider the setup in Figure 6.3 with

$$P(s) = \frac{6-s}{s^2+5s+6}$$

Give an upper bound for how fast the closed-loop system can be made, that is, give a value a such that the following specification is impossible to satisfy if $c > a$:

$$|S(i\omega)| \leq \frac{2\omega}{\sqrt{\omega^2+c^2}} \qquad \omega \in \mathbb{R}^+$$

Solutions to Exercise 6. Fundamental Limitations

6.1 **a.** The transfer function of the process $P(s)$ is given by

$$P(s) = \frac{s}{s^2 + 2s + 1}$$

and the zero is located in the origin.

b. The sensitivity function is given by $S(s) = \dfrac{1}{1 + P(s)C(s)}$. For $\omega = 0$ we will have $|S(i\omega)| = 1$ since $P(0) = 0$. Note that you can not cancel the process zero in $s = 0$ with your controller since you then would not have an asymptotically stable system.

c. The error $e(t)$ is given by $r(t) - \theta(t)$ and the static error is then given by the final value theorem, which can be used if all poles of $sE(s)$ have a strictly negative real-part.

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

Here the transfer function from r to e is given by:

$$G_{re}(s) = \frac{1}{1 + P(s)C(s)}$$

The following result is obtained if $r(t)$ is assumed to be a step, $R(s) = a/s$.

$$\lim_{s\to 0} sE(s) = \lim_{s\to 0} sG_{re}(s)R(s) = a$$

since $P(0) = 0$ (and thereby $G_{re}(0) = 1$). This means that the ball will not follow a reference trajectory that changes step-wise; there will be a static error equal to a. Hence, no matter the reference value, the ball will end up at the bottom of the cylinder.

An alternative explanation is that the sensitivity function S is 1 at $s = 0$, therefore

$$T(0) = \frac{P(0)C(0)}{1 + P(0)C(0)} = 1 - S(0) = 0$$

and then $y(t)$ does not follow $r(t)$ in stationarity.

d. The transfer function for the open loop with a PI controller is given by:

$$P(s)C(s) = \frac{s}{s^2 + 2s + 1} K \frac{s + 1/T_i}{s} = K \frac{s + 1/T_i}{s^2 + 2s + 1}$$

Here the process zero is canceled by the controller.

If we would have no stationary error the control signal from a PI controller would be constant. But if we have a constant control signal ω then that would imply that $\dot{\omega} = 0$, which would give $\theta = 0$ and we would get a stationary error (unless $r(t) = 0$). This is contradictory and therefore we must have a stationary error.

To remove the stationary error in this case, the controller would need a double integrator. It would also require that there are no control signal limitations, since u has to grow linearly to keep the ball at any stationary angle $\theta \neq 0$.

6.2 The sensitivity function is given by:

$$S(s) = \frac{1}{1 + P(s)C(s)}$$

From this it follows that

$$S(i\omega) = \frac{1}{1 + P(i\omega)C(i\omega)}$$

a. In the case of a purely imaginary process *pole* in $i\omega_p$ we have

$$P(i\omega_p) = \infty$$

and consequently

$$S(i\omega_p) = 0$$

b. A measurement disturbance n with frequency ω_p will have a vanishing effect on y and e, since

$$S(i\omega_p) = 0.$$

Note that this implies that n will have a big impact on z since $z(t) = y(t) - n(t)$.

c. No stabilizing controller can change the fact that $S(i\omega_p) = 0$. Cancellations of poles on the imaginary axis should always be avoided.

d. In the case of a purely imaginary process *zero* in $i\omega_z$ we have

$$P(i\omega_z) = 0$$

and consequently

$$S(i\omega_z) = 1$$

e. The transfer function from n to z is given by $-T(s)$, where T is the complementary sensitivity function. Since $S(i\omega_z) = 1$ and $S + T = 1$ it must hold that $T(i\omega_z) = 0$, i.e. an output disturbance with frequency ω_z will have no effect on z.

6.3 **a.** The transfer function from n to z is given by

$$G_{zn}(s) = -\frac{P(s)C(s)}{1 + P(s)C(s)}$$

We want to determine $C(s)$ such that $G_{zn}(s) = 5/(s+5)$. This gives the equation

$$-\frac{P(s)C(s)}{1 + P(s)C(s)} = \frac{5}{s+5} \Longrightarrow C(s) = -\frac{\frac{5}{s+5}}{P(s) \cdot (1 + \frac{5}{s+5})}$$

Inserting $P(s) = (3-s)/(s+1)^2$, we obtain

$$C(s) = -\frac{5\cdot(s+1)^2}{(3-s)(s+10)}$$

However, this is not a stabilizing controller. For example, the transfer function from n to u, $G_{un} = -\frac{C(s)}{1+P(s)C(s)}$, will be unstable because of the cancellation of the unstable zero in $P(s)$.

b. The specification

$$|S(i\omega)| \le \frac{2\omega}{\sqrt{\omega^2+36}} \qquad \omega \in \mathbb{R}^+$$

is equivalent to

$$\sup_\omega \left| \frac{\sqrt{\omega^2+36}}{2\omega} S(i\omega) \right| \le 1$$

However, $W_a(i\omega) = \frac{i\omega + a}{2i\omega}$ gives $|W_a(iw)| = \frac{\sqrt{\omega^2+36}}{2\omega}$, for $a = 6$ so the specification can equivalently be written

$$\sup_\omega |W_a(i\omega)S(i\omega)| \le 1$$

This makes the specification impossible to satisfy unless $|W_a(z)| \le 1$. We see here that $|W_a(3)| = |\frac{3+6}{2\cdot 3}| = \frac{3}{2} > 1$, so the specification is impossible to satisfy for $a = 6$.

c. The Bode plot of $P(s)$ is given in Figure 6.1 and the sensitivity function when $C(s) = 1$ is given in Figure 6.2 together with the specification. Since the specification $\frac{2\omega}{\sqrt{\omega^2+1}} = 0$ when $\omega = 0$ the controller $C(s)$ must contain an integrator. To avoid instability we must also lift the phase curve through adding a zero and decrease the gain in the open-loop, $P(s)C(s)$. A controller on the form

$$C(s) = K\cdot\frac{s/b+1}{s}$$

with e.g. $K = 0.17$, $b = 0.5$ will do the job.

To plot the specification on top of the Bode plot of S the following Matlab commands can be used:

```
>>[mag, fas, w] = bode(S);
>>loglog(w, 2.*w./sqrt(w.^2+1), 'r --')
>>hold on
>>bode(S)
```

6.4 The first case is impossible, because there is a time-delay of 2 seconds in the plant, so the control signal will affect the output with this delay. Thus, the controller would need to be non-causal to achieve the specification.

The second specification in the figure says that the gain should be below 2 (actually the requirement is closer to ≈ 1.6).

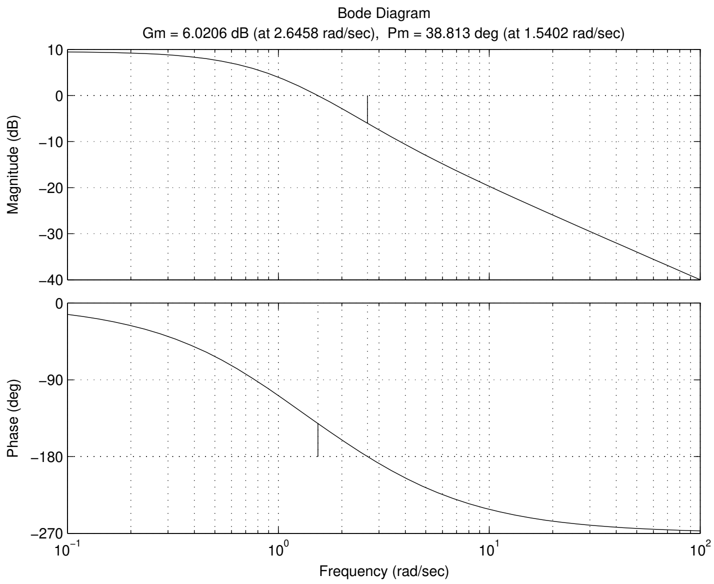

Figure 6.1 Bode plot of $P(s)$ in Problem 6.3.

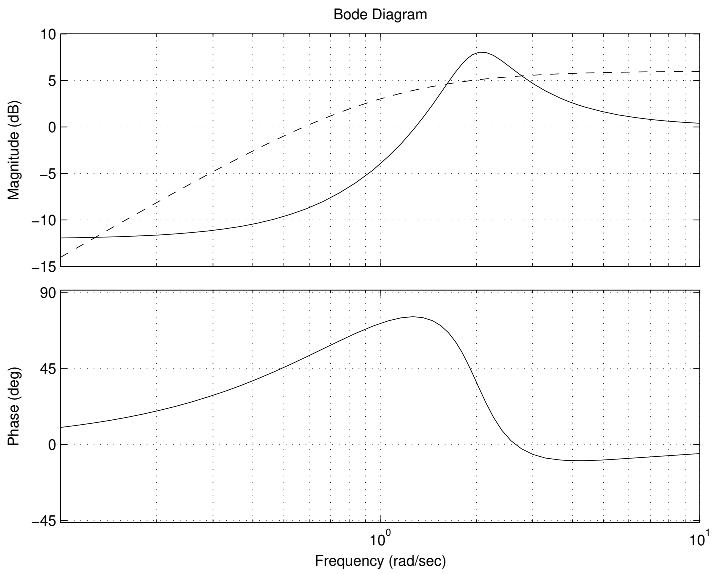

Figure 6.2 Sensitivity function when $C(s) = 1$ and the specification (dashed) in Problem 6.3.

From the lecture notes we know that if there is both an unstable pole p and an unstable zero z, we have the following fundamental limitation:

$$\|S\|_\infty \geq \left| \frac{z+p}{z-p} \right|$$

We know that $S + T = 1$, so in this case, we have

$$\|T\|_\infty = \|S-1\|_\infty \geq \|S\|_\infty - 1 \geq \left| \frac{20+10}{20-10} \right| - 1 = 2$$

where the first inequality follows from the reverse triangle inequality

$$\big| \|x\| - \|y\| \big| \leq \|x-y\|$$

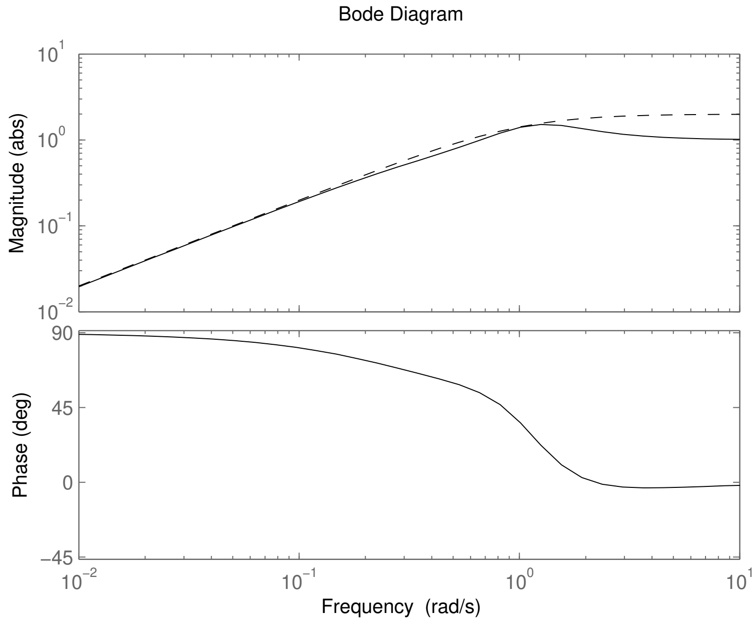

Figure 6.3 Sensitivity function for the proposed controller and the specification (dashed) in Problem 6.3.

It is also possible to get to the same conclusion without using the unstable zero, since the existence of a fast unstable pole is enough the make the specification impossible to achieve.

The third case is possible. If proportional control, $C(s) = K$ is used the closed-loop transfer function becomes $G(s) = \frac{K}{s-3+K}$. For stability it is required that $K > 3$. The static gain is given by $\frac{K}{K-3}$. Since it is a first order system there will be no overshoot in the step response, which means that a P-controller with $K > 6$ will fulfill the specification to stay in the interval $[0, 2]$.

6.5 **a.** Assume $\sup_\omega |W_S(i\omega)S(i\omega)| \leq 1$ and $\sup_\omega |W_T(i\omega)T(i\omega)| \leq 1$ are satisfied.

We know that $1 = |S(s_0) + T(s_0)| \leq |S(s_0)| + |T(s_0)|$ (triangle inequality).

If $|W_S(s_0)| > 2$ for some right half place s_0, then $|S(s_0)| < 1/2$, since

$$\sup_\omega |W_S(i\omega)S(i\omega)| = \sup_{Re(s)\geq 0} |W_S(s)S(s)| \leq 1 \text{(Maximum Modulus Theorem).}$$

Analogously we get $|T(s_0)| < 1/2$. Then

$$1 = |S(s_0) + T(s_0)| \leq |S(s_0)| + |T(s_0)| < 1$$

and we arrive to contradiction. Hence either $|W_T(s)T(s)| > 1$ or $|W_S(s)S(s)| > 1$ and the corresponding specification must fail.

b. We have

$$W_S(1) = \left(\frac{1+0.1}{1}\right)^n = \left(\frac{1+10}{10}\right)^n = W_T(1)$$

and the value is larger than 2 for $n \geq 8$. Hence, the statement in **a** shows that the specifications are incompatible.

6.6 **a.** The requirements on $|S(i\omega)| = \bar{\sigma}(S(i\omega))$ and $|T(i\omega)| = \bar{\sigma}(T(i\omega))$ may be formulated as

$$|S(i\omega)| \le \tfrac{1}{10}, \quad \omega \le 0.1, \qquad |T(i\omega)| \le \tfrac{1}{10}, \quad \omega \ge 2$$
$$|S(0)| \le \tfrac{1}{100}$$

b. The specifications in **a** can be formulated with weighting functions W_S and W_T as

$$|S(i\omega)| \le |W_S^{-1}(i\omega)|, \quad \forall\omega$$
$$|T(i\omega)| \le |W_T^{-1}(i\omega)|, \quad \forall\omega$$

If e.g. W_S^{-1} and W_T^{-1} are chosen according to

$$W_S^{-1}(s) = a_1\left(1 + \frac{s}{b_1}\right), \qquad W_T^{-1}(s) = \frac{a_2}{s}\left(1 + \frac{s}{b_2}\right)$$

we get

$$W_S^{-1}(s) = \frac{1}{100}(1 + 100s), \qquad W_T^{-1}(s) = \frac{0.14}{s}\left(1 + \frac{s}{2}\right)$$

6.7 The specification

$$|S(i\omega)| \le \frac{2\omega}{\sqrt{\omega^2 + c^2}} \qquad \omega \in \mathbb{R}^+$$

is equal to

$$\sup_\omega \left| \frac{\sqrt{\omega^2 + c^2}}{2\omega} S(i\omega) \right| \le 1$$

Since

$$W_S(i\omega) = \frac{i\omega + c}{2i\omega}$$

gives

$$|W_S(i\omega)| = \frac{\sqrt{\omega^2 + c^2}}{2\omega}$$

the specification can be written

$$\sup_\omega |W_S(i\omega)S(i\omega)| \le 1$$

This specification is impossible to meet when the process has a RHP zero in $s = z$, unless $|W_S(z)| \le 1$. Here we have a zero in $z = 6$, so we must have

$$\frac{6 + c}{12} \le 1 \Leftrightarrow c \le 6 = a.$$