FRTN10 Exercise 2. Stability and Robustness

2.1 **a.** Analyze the stability of the system to the left in Figure 2.1 using the Small Gain Theorem. For what values of the gain $\|\Delta\|$ can stability be guaranteed?

b. Analyze the stability of the system to the right in Figure 2.1, where K is a constant feedback gain. For what values of the gain K can stability be guaranteed? Explain why the result in different from that of **a**.

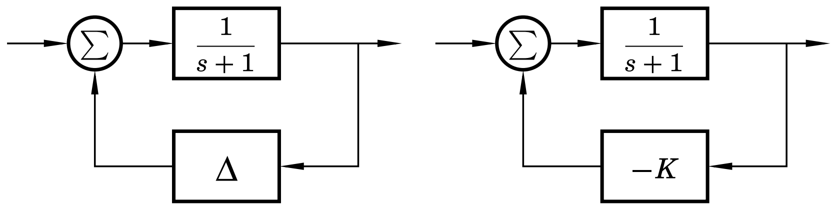

Figure 2.1 Systems in Problem 2.1.

2.2 Consider the system in Figure 2.2.

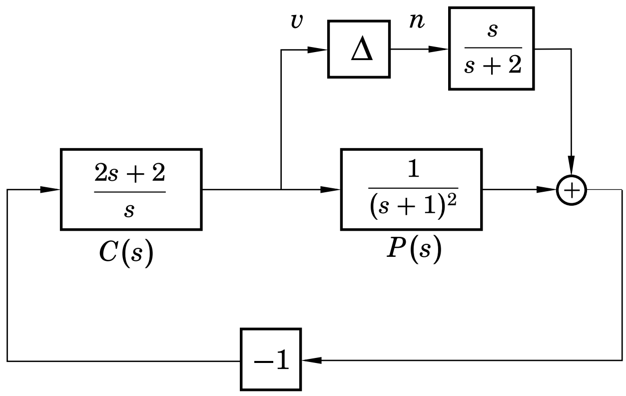

Figure 2.2 System in Problem 2.2.

a. Setting $\Delta = 0$, find the transfer function $G_{vn}(s)$ and show that it is stable.

b. Determine the largest gain $\|\Delta\|$ for which the closed-loop system is guaranteed to the stable. The Bode diagram of $G_{vn}(s)$ is provided in Figure 2.3.

c. The Δ block is used to account for uncertainty in the process model. Explain the role of the factor $\frac{s}{s+2}$ multiplying Δ.

2.3 Consider the system in Figure 2.4, where Δ is an unknown dynamical system, and G is a static TITO system

$$G = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}$$

For what values of the gain $\|\Delta\|$ is the closed loop guaranteed to be stable?

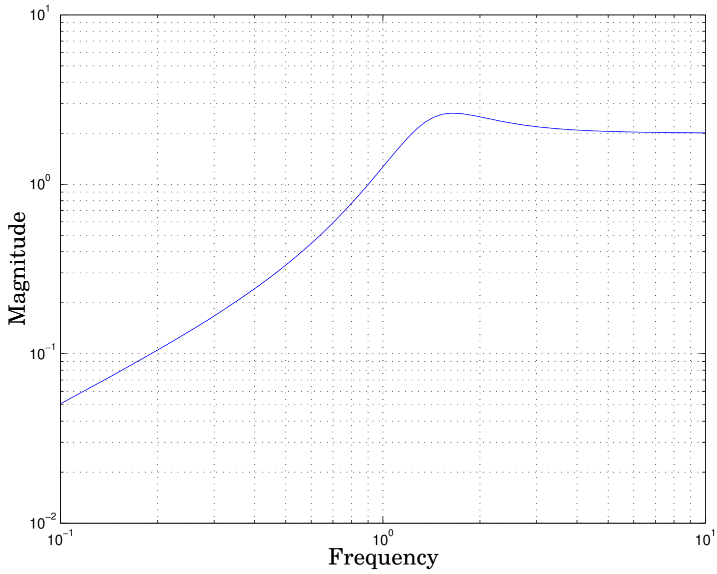

Figure 2.3 Bode magnitude diagram for $G_{vn}(s)$ in Problem 2.2.

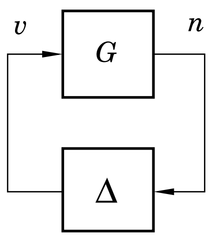

Figure 2.4 System in Problem 2.3.

2.4 Consider the system

$$G(s) = \begin{bmatrix} \frac{1}{s+2} & -\frac{1}{s+2} \\ \frac{1}{s+2} & \frac{s+1}{s+2} \end{bmatrix}$$

A singular value plot of $|G(i\omega)|$ is given in Figure 2.5. What is the $\mathcal{L}_2$ gain of the system, for what frequency is it attained, and what are the corresponding input and output directions?

2.5 Consider the feedback system in Figure 2.6, with

$$C(s) = 1.4\frac{s+1}{s}, \quad P(s) = \frac{1}{(s+1)^2}$$

a. Calculate the transfer functions $G_{zv}(s) = S(s)$ (the sensitivity function) and $-G_{zn}(s) = T(s)$ (the complementary sensitivity function).

b. Figure 2.7 shows gain curves for the sensitivity function and the complementary sensitivity function. Which curve represents which function?

c. In what frequency region is there good tracking of the reference value? In what frequency region is there good attenuation of measurement noise?

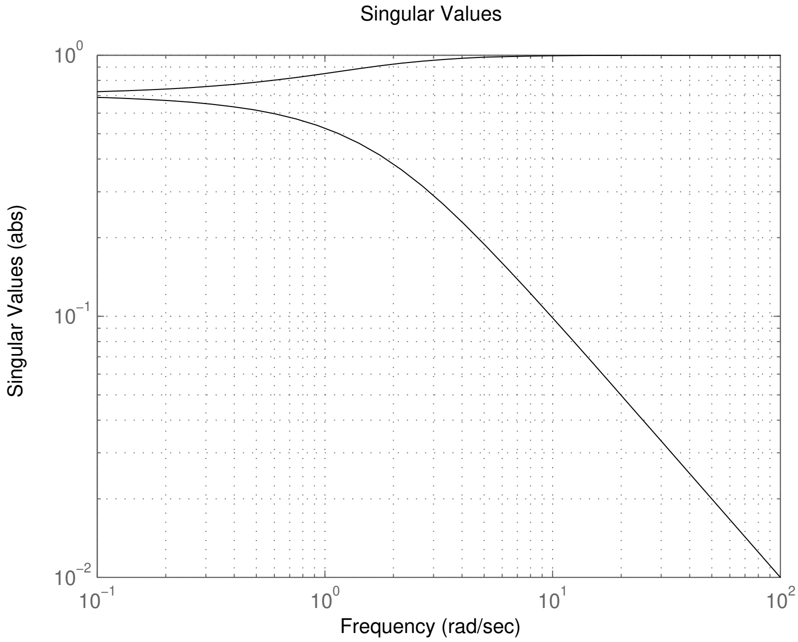

Figure 2.5 Singular value plot in Problem 2.4.

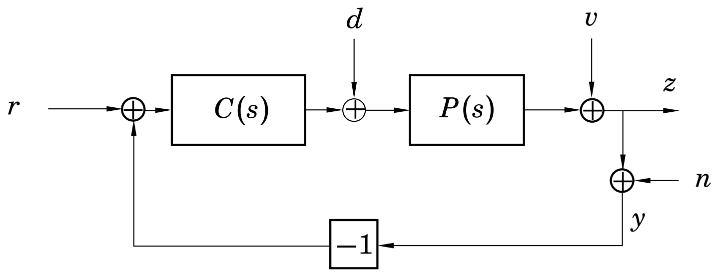

Figure 2.6 Feedback system in Problems 2.5 and 2.6.

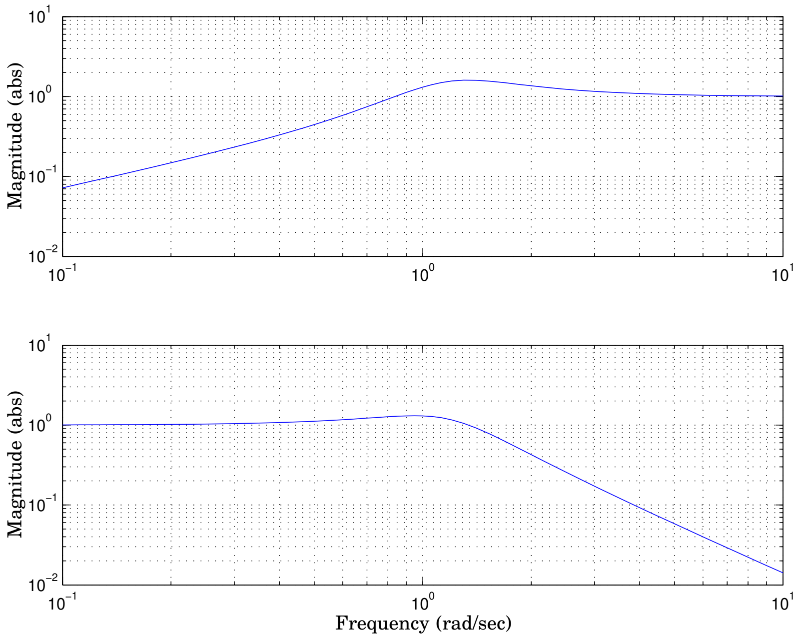

Figure 2.7 The two magnitude curves in Problem 2.5.

2.6 Again consider the feedback system in Figure 2.6, where the process, $P(s)$, is now given by

$$P(s) = \frac{1}{(s+1)(s+2)}$$

Three different controllers were designed,

$$C_1(s) = 10 \qquad C_2(s) = 10\frac{s+1}{s} \qquad C_3(s) = 10\frac{s+1}{s}e^{-0.1s}$$

where the last one has a small delay.

a. Figure 2.8 shows sensitivity functions, corresponding to the three different control designs C_1–C_3. Match the controllers C_1–C_3 with the sensitivity functions A–C.

b. Figure 2.9 shows responses to a step output disturbance, v, corresponding to the three different control designs C_1–C_3. Match the controllers C_1–C_3 with the step responses I–III.

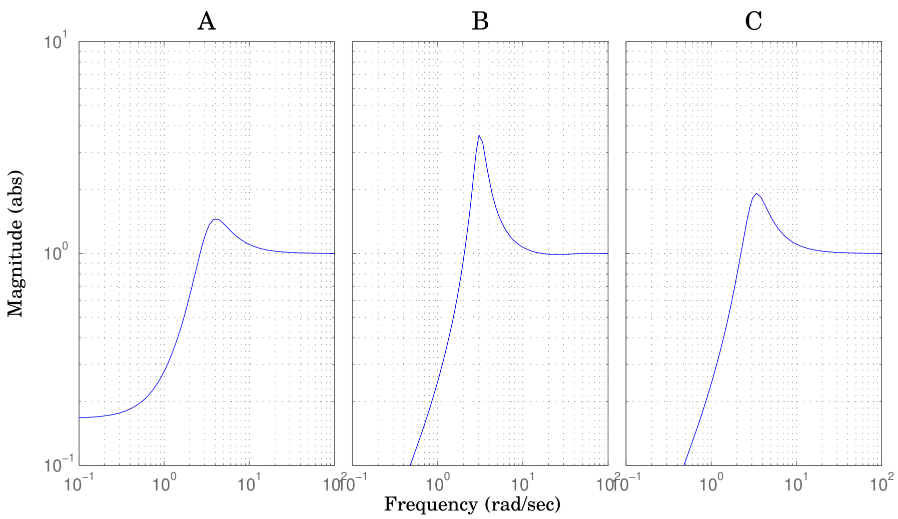

Figure 2.8 Sensitivity functions in Problem 2.6.

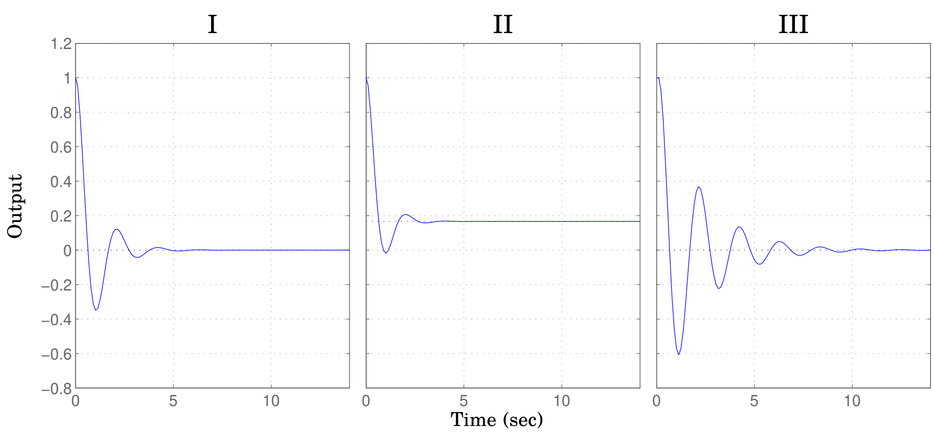

Figure 2.9 Load disturbance step responses in Problem 2.6.

2.7 (*) Consider a water tank with a separating wall. The wall has a hole at the bottom, as can be seen in Figure 2.10.

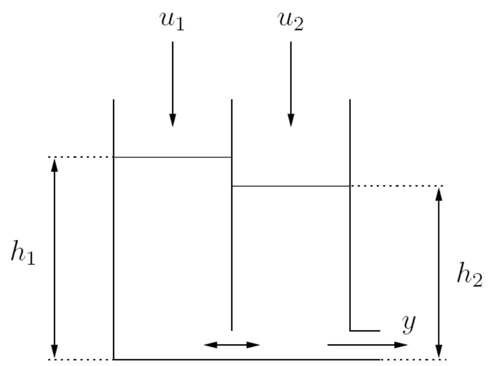

Figure 2.10 Water tank in Problem 2.7.

The input signals are the inflows of water to the left, u_1 [cm^3/s], and the right, u_2 [cm^3/s]. The water levels are denoted by h_1 [cm] and h_2 [cm], respectively. The outflow y [cm^3/s] is proportional to the water level in the right half of the tank:

$$y(t) = \alpha h_2(t)$$

The flow between the tank halves is proportional to the difference in level:

$$f(t) = \beta(h_1(t) - h_2(t))$$

(positive flow from left to right)

The signals h_i, u_i and y are thought of as deviations from a linearization point, and may therefore be negative. Assume that the two tank halves each have area $A_1 = A_2 = 1$ [cm^2].

a. Write the system in state-space form.

b. What is the transfer matrix from $\begin{pmatrix} u_1 & u_2 \end{pmatrix}^T$ to y?

c. Assume that $\alpha = \beta = 1$. A sigma plot of the singular values of the system reveals that the maximum gain is attained for frequency 0. Using this information, calculate the L_2 gain of the system.

d. It turns out that the L_2 gain is larger than one. How is this possible? Can there be more water coming out from the tank than what is poured into it? Explain what is wrong with this reasoning!

Solutions to Exercise 2. Stability and Robustness

2.1 Let $G(s) = \dfrac{1}{s+1}$.

a. The system is guaranteed to be stable according to the Small Gain Theorem if $\|G\| \cdot \|\Delta\| < 1$. We see that $\|G\| = \sup_\omega |G(i\omega)| = 1$, so the system is guaranteed to be stable if $\|\Delta\| < 1$.

b. The transfer function of the closed-loop system is

$$\frac{G(s)}{1+G(s)K} = \frac{1}{s+1+K}$$

The pole is located in $s = -(1+K)$, so the system is stable for any gain $K > -1$. We can compare this to the result in **a**, which only guarantees that the system is stable when the gain $\|\Delta\| < 1$.

The different results arise from the fact that the Small Gain Theorem is conservative in nature, i.e., it gives a *sufficient* condition on stability, but that condition may not be *necessary*. The theorem makes no *a priori* assumptions on Δ, which may be any system and not just a constant scalar as in **b**. Studying at the closed-loop poles of a known LTI system, on the other hand, shows exactly when the system is stable.

2.2 **a.** We have

$$C(s) = \frac{2s+2}{s} \qquad P(s) = \frac{1}{(s+1)^2} \qquad W(s) = \frac{s}{s+2}$$

$$G_{vn}(s) = -\frac{C(s)W(s)}{1+P(s)C(s)} = -\frac{2s^4+6s^3+6s^2+2s}{s^4+4s^3+7s^2+8s+4} = -\frac{2s^3+4s^2+2s}{s^3+3s^2+4s+4}$$

The characteristic polynomial s^3+3s^2+4s+4 has LHP roots since all coefficients are positive and $3 \cdot 4 > 4$.

b. The peak magnitude in the Bode diagram shows $\|G_{vn}\| \approx 2.6$. The Small Gain Theorem shows stability for all perturbations Δ satisfying

$$\|\Delta\| \cdot \|G_{vn}\| < 1 \quad \Rightarrow \quad \|\Delta\| < 0.38$$

c. Process models are typically more accurate in the low-frequency range. As the frequency increases there is usually higher-order dynamics and non-linearities in the real process, which are not covered by the model. The high-pass filter $\frac{s}{s+2}$ in this example is used to indicate that the uncertainty is small for low frequencies (below approximately $\omega = 2$).

2.3 The gain of G is given by its largest singular value, $\overline{\sigma}$.

$$G^*G = \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix}$$

$$\det(\lambda I - G^*G) = \det\begin{bmatrix}\lambda-4 & -6\\ -6 & \lambda-13\end{bmatrix} = \lambda^2 - 17\lambda - 16 = (\lambda-1)(\lambda-16)$$

$$\overline{\sigma} = \sqrt{16} = 4$$

The Small Gain Theorem guarantees stability for any Δ with $\|\Delta\| \cdot \|G\| < 1$, which gives $\|\Delta\| < 1/4$.

2.4 From the sigma plot we see that the maximum of the largest singular value is 1, attained when $\omega \to \infty$. The transfer matrix then becomes

$$\lim_{\omega\to\infty} G(i\omega) = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$

From this it is clear that the input and output directions are both $\begin{bmatrix}0\\ 1\end{bmatrix}$.

2.5 **a.**

$$S(s) = \frac{1}{1+CP} = \frac{s^3+2s^2+s}{s^3+2s^2+2.4s+1.4} =$$
$$= \frac{(s+1)(s^2+s)}{(s+1)(s^2+s+1.4)} = \frac{s^2+s}{s^2+s+1.4}$$
$$T(s) = \frac{CP}{1+CP} = \frac{1.4(s+1)}{s^3+2s^2+2.4s+1.4} = \frac{1.4}{s^2+s+1.4}$$

b. The sensitivity is typically small for low frequencies (where the feedback suppresses disturbances and model errors) and close to 1 for high frequencies (where the feedback gain is close to 0). The opposite holds for the complementary sensitivity. The upper curve is hence $S(s)$ and the lower curve $T(s)$.

Since the transfer functions are given, we could also use direct calculations to show that $S(0) = 0$, $S(\infty) = 1$, $T(0) = 1$, $T(\infty) = 0$.

c. The output z tracks the reference r well if $G_{zr}(i\omega) = T(i\omega)$ is close to 1. This is the case for approximately $\omega < 1$. Measurement noise n is attenuated well if $G_{zn}(i\omega) = -T(i\omega)$ is close to 0. This is the case for approximately $\omega > 1$. (Note the design trade-off between closed-loop bandwidth and attenuation of measurement noise.)

2.6 **a.** The sensitivity function is given by $S = \frac{1}{1+PC}$, so S is small at frequencies where PC is large. The stationary gain of P is finite. C_2 and C_3 both have integral action and infinite stationary gain. Thus, for these controllers, S will go to zero as $\omega \to 0$. C_1, being a pure P-controller, has a finite stationary gain. S will then also have a finite stationary gain.

C_2 and C_3 are PI-controllers, but C_3 has a delay which will introduce extra phase lag. This decreases the phase margin and therefore introduces a higher sensitivity peak. Thus, we have: $C_1 \to A$, $C_2 \to C$, and $C_3 \to B$.

b. Since controller C_1 does not have integral action, we will get a stationary error in the response to a constant output disturbance, v. The response using the delayed controller C_3 will be less damped than the response using the PI-controller because of the smaller phase margin, C_2. This gives: $C_1 \to$ II, $C_2 \to$ I, and $C_3 \to$ III.

2.7 **a.** If V_i is the water volume in tank i, then it follows from mass balance that

$$\dot{V}_1 = A_1\dot{h}_1 = (u_1 - f), \qquad \dot{V}_2 = A_2\dot{h}_2 = (u_2 + f - y),$$

i.e., with $h = [\,h_1 \quad h_2\,]^T$ it follows that

$$\dot{h} = \begin{pmatrix} -\frac{1}{A_1}\beta & \frac{1}{A_1}\beta \\ \frac{1}{A_2}\beta & -\frac{1}{A_2}(\beta+\alpha) \end{pmatrix} h + \begin{pmatrix} \frac{1}{A_1} & 0 \\ 0 & \frac{1}{A_2} \end{pmatrix} u$$
$$y = (\,0 \quad \alpha\,)\, h$$

b.

$$G(s) = \frac{1}{s^2 + (2\beta+\alpha)s + \alpha\beta}\,(\,\alpha\beta \quad \alpha(s+\beta)\,)$$

c. For $\omega = 0$ we have the static gain

$$G(0) = [\,1 \quad 1\,]$$

The L_2 gain is given by the largest singular value of $G(0)$:

$$\det(\lambda I - G^*(0)G) = 0 \quad\Rightarrow\quad \lambda_1 = 0, \lambda_2 = 2 \quad\Rightarrow\quad \overline{\sigma} = \sqrt{2}$$

d. For $\omega = 0$, our system is given by $y = u_1 + u_2$. While it holds that the total mass flow is equal,

$$\int_0^\infty (u_1 + u_2)\, dt = \int_0^\infty y\, dt$$

this does NOT imply that the L_2 norms of the input and output signals are equal. Taking for instance

$$u_1(t) = u_2(t) = \begin{cases} 1 & \text{if } t \le T \\ 0 & \text{if } t > T \end{cases}$$

the L_2 norm of the input is

$$\|u\| = \sqrt{\int_0^T (u_1^2 + u_2^2) dt} = \sqrt{2T}$$

while the L_2 norm of the output is $\sqrt{2}$ times larger:

$$\|y\| = \sqrt{\int_0^T (u_1 + u_2)^2 dt} = \sqrt{4T}$$