FRTN10 Exercise 1. Control Basics

If you master the basic control systems theory, including LTI system representations, Bode diagrams, block diagrams, and stability, you can skip straight to Problem 1.9.

LTI System Representations

1.1 For each of the linear, time-invariant systems below,

- find the transfer function $G(s)$.
- calculate the poles and zeros of $G(s)$.
- determine whether the system is (asymptotically) stable. If so, calculate the static gain of the system.
- introduce state variables x and write the system in state-space form.

a. $T\dot{y}(t) + y(t) = u(t)$

b. $\ddot{y}(t) + 2\dot{y}(t) + 4y(t) = \dot{u}(t)$

c. $\ddot{y}(t) + 3\dot{y}(t) = u(t-2)$

1.2 Figure 1.1 shows the step response of four linear systems A–D. Give the structure of the corresponding transfer function $G_A(s)$–$G_D(s)$. For first-order systems, you should write down the exact transfer function, including the values of all coefficients.

Bode Diagrams

1.3 A lag filter has the Bode diagram shown in Figure 1.2.

a. How much are input signals of frequencies 0, 10, and ∞ rad/s amplified, respectively?

b. Estimate the output $y(t)$ if the input is $u(t) = 0.5\sin\left(100t - \frac{\pi}{2}\right),\ -\infty < t < \infty$.

c. Estimate the transfer function $G(s)$ of the system.

1.4 Sketch the Bode diagram (amplitude and phase) of the following systems:

a. $G(s) = \dfrac{1}{1+sT},\quad T > 0$ (a low-pass filter)

b. $G(s) = K\left(1 + \dfrac{1}{sT_i}\right),\quad K, T_i > 0$ (a PI controller)

c. $G(s) = \dfrac{N(s+b)}{s+bN},\quad b > 0,\ N > 1$ (a lead filter)

Block Diagrams

1.5 Given the block diagram in Figure 1.3, calculate

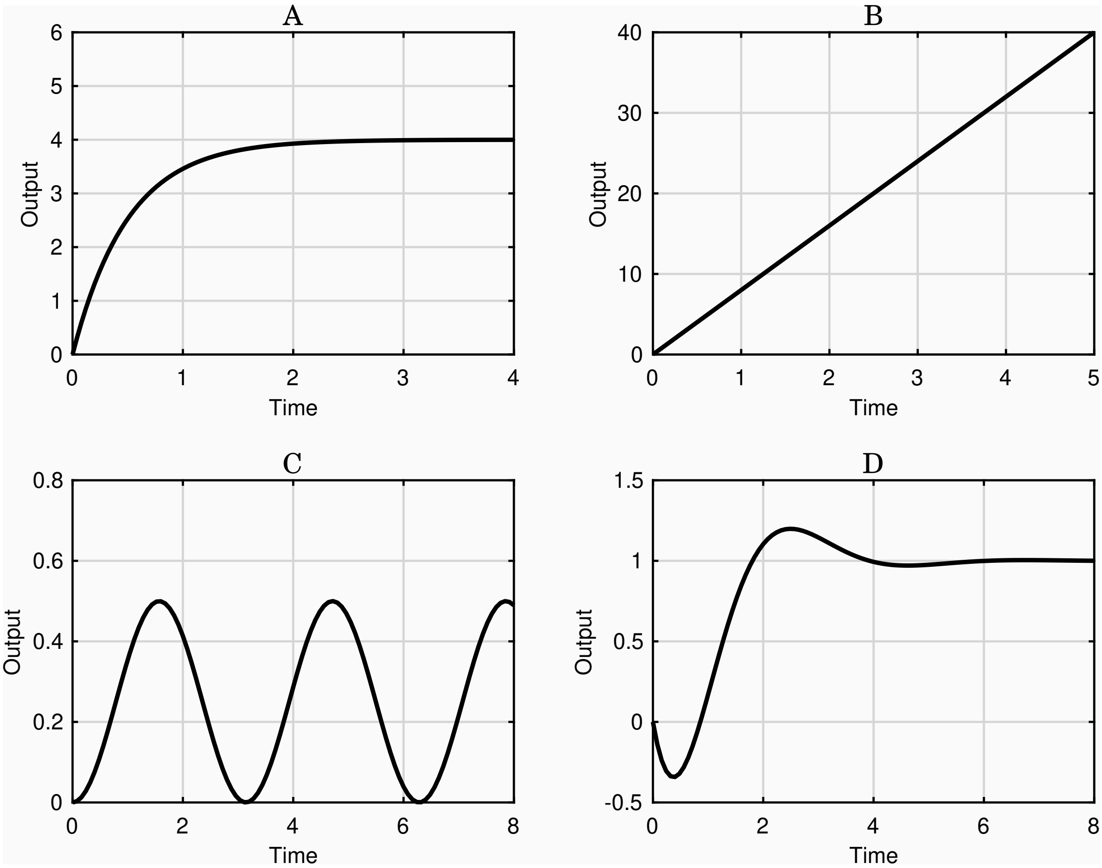

Figure 1.1 Step responses of four linear systems.

a. the transfer function $G_{yr}(s)$.

b. the transfer function $G_{vw}(s)$.

1.6 Given the block diagram in Figure 1.4, calculate

a. the transfer function $G_{yr_y}(s)$.

b. the transfer function $G_{u_1 r_{u_1}}(s)$.

Stability

1.7 Determine whether the following systems are stable or unstable.

a. $\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x$

b. $\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 3 & -2 & 0 \\ 7 & -3 & 3 \end{bmatrix} x$

c. $G(s) = \dfrac{s-1}{s^2 + 1.3s + 4.7}$

d. $G(s) = \dfrac{1}{s^4 - 2s^3 + 5s^2 + 6s + 1}$

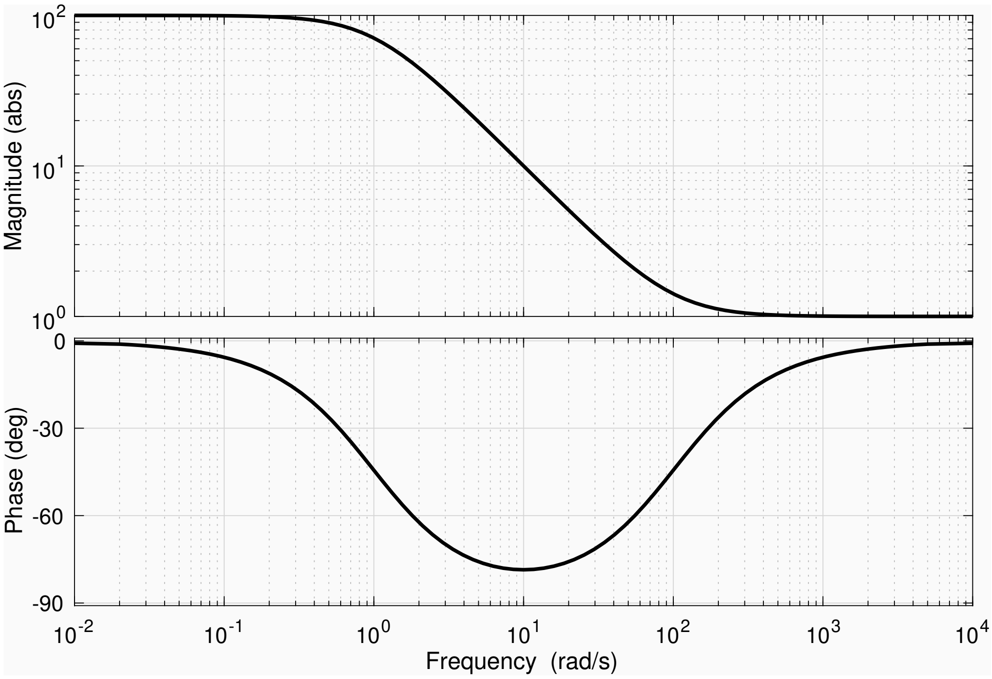

Figure 1.2 Bode diagram

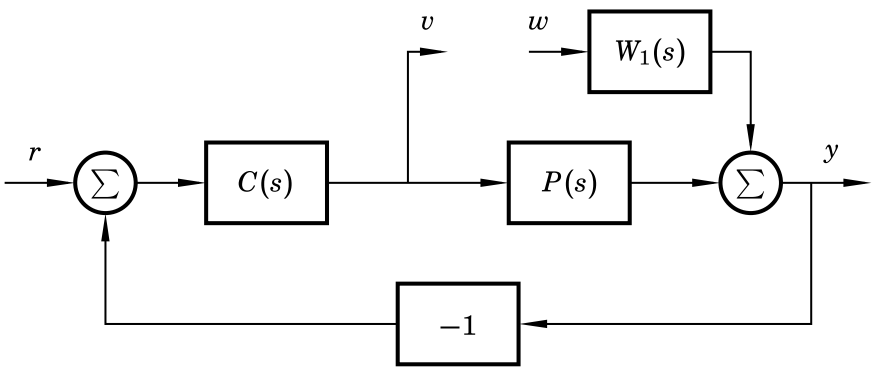

Figure 1.3 Block diagram in Problem 1.5.

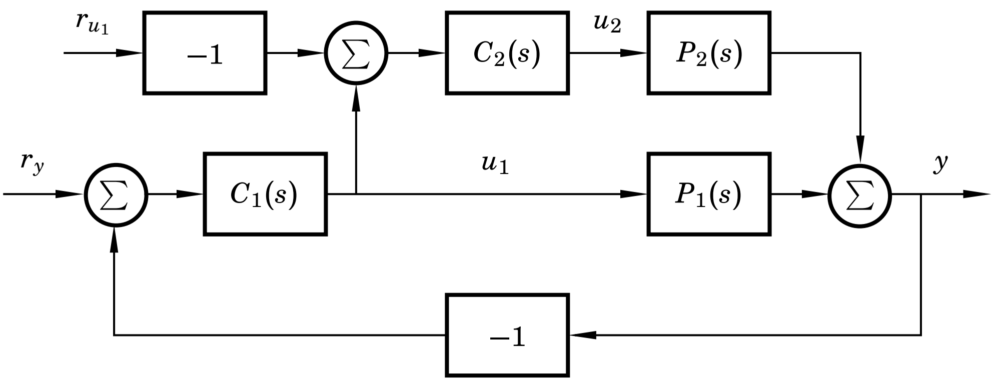

Figure 1.4 Block diagram in Problem 1.6.

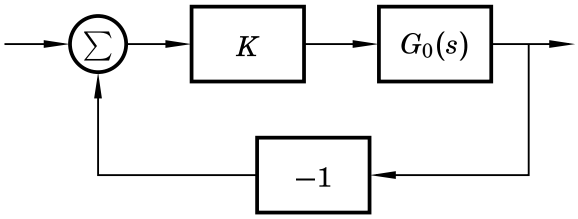

Figure 1.5 Simple control loop

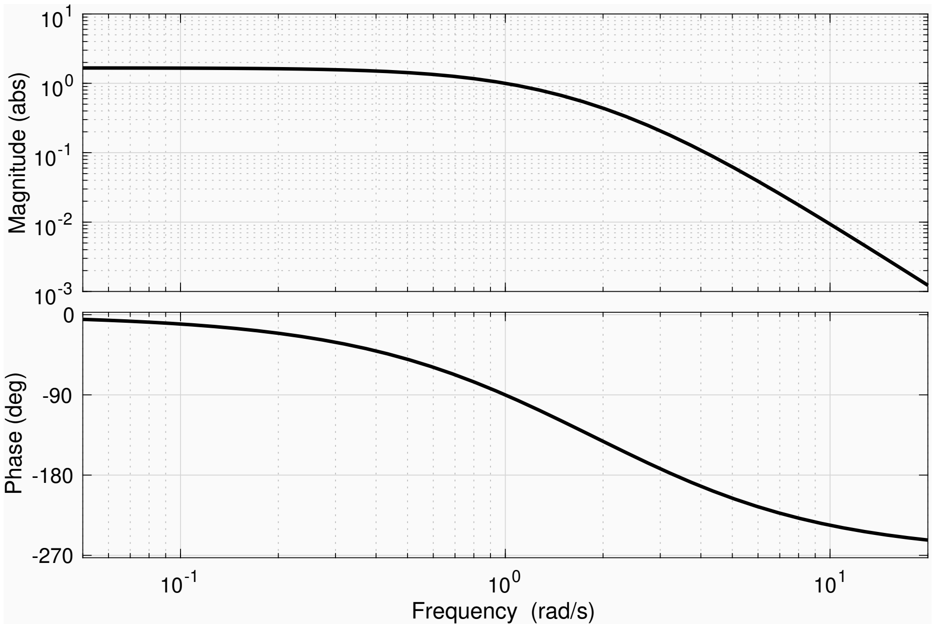

Figure 1.6 Bode diagram

1.8 Consider the simple feedback loop in Figure 1.5, where

$$G_0(s) = \frac{10}{(s+1)(s+2)(s+3)}$$

and $K > 0$.

a. For what values of K is the closed loop stable? Use direct calculation of the closed-loop characteristic polynomial.

b. For what values of K is the closed loop stable? Use the Nyquist theorem and the Bode diagram of $G_0(s)$ in Figure 1.6.

Multivariable LTI System Representations

1.9 Consider the multivariable process in Figure 1.7.

a. Find the transfer matrix of the process.

b. Introduce state variables and write the system in state-space form.

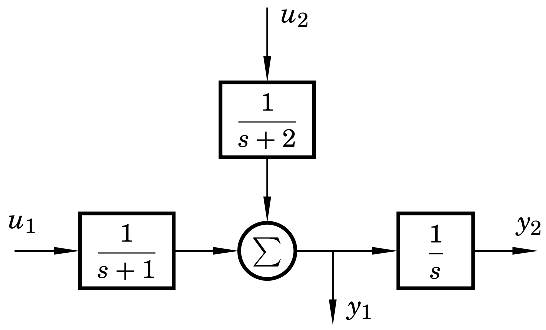

Figure 1.7 System in Problem 1.9

1.10 A system is given by

$$\begin{aligned}\dot{x}_1 &= -2x_1 + x_2 + u_1 \\ \dot{x}_2 &= -3x_2 + u_1 + 2u_2 \\ y_1 &= x_1 + x_2 \\ y_2 &= 2x_1 + u_1 \\ y_3 &= 2x_2 + u_2\end{aligned}$$

a. Find the system matrices A, B, C, and D.

b. Calculate the transfer matrix $G(s)$.

1.11 A system with two inputs and one output is modeled by the differential equation

$$\ddot{y} + a_1\dot{y} + a_2 y = b_{11}\dot{u}_1 + b_{12}u_1 + b_{21}\dot{u}_2 + b_{22}u_2.$$

a. Find the transfer matrix.

b. **(*)** Express the system in state-space form.

Solutions to Exercise 1. Control Basics

1.1 **a.** Taking the Laplace transform of the differential equation ($\dot{x}(t) \Leftrightarrow sX(s)$ if ignoring initial values), we obtain

$$TsY(s) + Y(s) = U(s) \quad \Leftrightarrow \quad Y(s) = \underbrace{\frac{1}{1+sT}}_{=G(s)} U(s)$$

The poles p are the roots of the denominator polynomial, $p = -1/T$ in this case. The zeros z are the roots of the numerator polymial, none in this case.

The system is stable if and only if (iff) all poles are in the left half-plane (LHP). In this case the system is stable iff $T \geq 0$.

If $T \geq 0$ then the static gain is given by $G(0) = 1$.

Introducing (for instance) $x = y$, we can write the system as

$$\begin{aligned} \dot{x} &= -\frac{1}{T}x + \frac{1}{T}u \\ y &= x \end{aligned}$$

b.

$$s^2Y(s) + 2sY(s) + 4Y(s) = sU(s) \quad \Leftrightarrow \quad Y(s) = \underbrace{\frac{s}{s^2+2s+4}}_{=G(s)} U(s)$$

There are two poles, $p_{1,2} = -1 \pm i\sqrt{3}$ and one zero, $z = 0$.

The system is stable since the poles are in the LHP. This can also be seen directly from the numerator polynomial, where all coefficients are positive (both sufficient and necessary condition for second-order polynomials).

The static gain is $G(0) = 0$.

Since the system has a zero, the simplest choice $x_1 = y$, $x_2 = \dot{y}$ does not work. Instead we can use one of the canonical forms, for instance the controllable canonical form (see the Collection of Formulae), yielding

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -4x_1 - 2x_2 + u \\ y &= x_2 \end{aligned}$$

c. Taking the Laplace transform of the time delay $\big(x(t-L) \Leftrightarrow X(s)e^{-sL}\big)$, we obtain

$$s^2Y(s) + 3sY(s) = U(s)e^{-2s} \quad \Leftrightarrow \quad Y(s) = \underbrace{\frac{e^{-2s}}{s^2+3s}}_{=G(s)} U(s)$$

Since the transfer function is not rational, we cannot calculate all of its poles and zeros. The time delay does not affect the stability or the static gain of the system, however. It has a pole in zero (an integrator), which is enough to conclude that the system is unstable. The system does not have a static gain (or, it can be said to have infinite gain).

Since the tranfer function is not rational, we cannot write the system in standard state-space form using a finite number of state variables.

1.2 The step response in A looks like a typical stable first-order system. The static gain is 4, and the step response has reached 63% of this at time 0.5, which is the system's time constant T. This gives the transfer function

$$G_A(s) = \frac{4}{1+0.5s}$$

The step response in B describes an integrator, i.e., a first-order system with a pole in the origin. The slope of the response is 8, which implies the transfer function

$$G_B(s) = \frac{8}{s}$$

The step response in C shows oscillations, which implies a second-order system with complex poles. Further, the oscillations are undamped (relative damping $\zeta = 0$). The transfer function has the structure

$$G_C(s) = \frac{a}{s^2+\omega^2} \qquad a, \omega > 0$$

The step response in D has an undershoot, which implies that the system has a zero in the right half-plane. The step response is oscillatory but damped, implying a second-order system with complex poles and relative damping $0 < \zeta < 1$. The transfer function has the structure

$$G_D(s) = \frac{a-s}{s^2+2\zeta\omega s+\omega^2} \qquad a, \omega > 0, \quad 0 < \zeta < 1$$

1.3 **a.** From the magnitude curve we read out 100 (low-frequency asymptote), 10, and 1 (high-frequency asymptote), respectively.

b. The stationary response to $u(t) = a\sin(\omega t + \phi)$ is given by

$$y(t) = |G(i\omega)| a \sin(\omega t + \phi + \arg G(i\omega))$$

From the Bode diagram at frequency $\omega = 100$ rad/s we read out $|G(i100)| \approx 1.5$ and $\arg G(i100) \approx -45° = -\pi/4$. We obtain

$$y(t) \approx 0.75\sin(100t - 3\pi/4)$$

c. We have already noted that $|G(i0)| = 100$ and $|G(i\infty)| = 1$. Around $\omega = 1$ the slope of the amplitude curve decreases by 1 and the phase decreases by almost 90°, and around $\omega = 100$ the slope increases by 1 and the phase increases by almost 90°. This indicates that the system has a pole in -1 and a zero in -100. We can conclude that

$$G(s) \approx \frac{s+100}{s+1}$$

1.4 **a.** The low-frequency asymptote ($s \to 0$) is given by $G(s) \approx 1$, and the high-freqency asymptote ($s \to \infty$) is given by $G(s) \approx \frac{1}{sT}$. The slope of the magnitude curve starts at 0 and decreases by 1 around $\omega = 1/T$ (the magnitude of the pole). The phase curve starts at 0° and decreases by 90° around $\omega = 1/T$. See Figure 1.1.

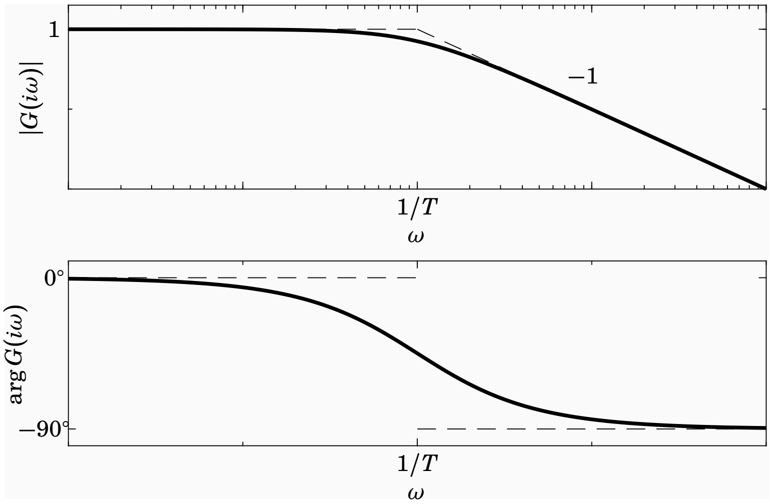

Figure 1.1 Bode diagram of $G(s) = \dfrac{1}{1+sT}$

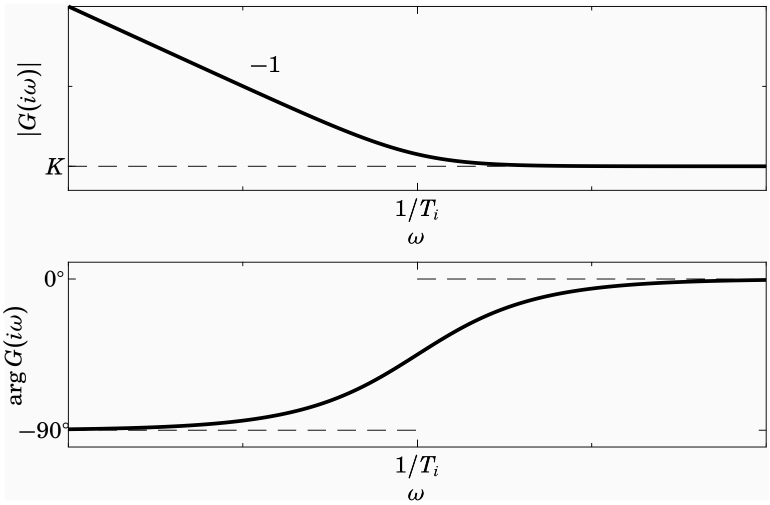

Figure 1.2 Bode diagram of $G(s) = K\Big(1 + \dfrac{1}{sT_i}\Big)$

b. Low-frequency asymptote: $G(s) \approx \frac{K}{sT_i}$ High-freqency asymptote: $G(s) \approx K$. The slope of the magnitude curve starts at -1 and increases by 1 around $\omega = 1/T_i$ (the magnitude of the zero). The phase curve starts at $-90°$ and increases by $90°$ around $\omega = 1/T_i$. See Figure 1.2.

c. Low-frequency asymptote: $G(s) \approx 1$. High-freqency asymptote: $G(s) \approx N$. The slope of the magnitude curve starts at 0, increases by 1 around $\omega = b$, and decreases by 1 around $\omega = bN$. The phase curve starts at $0°$, increases around $\omega = b$, and decreases down to $0°$ again at $\omega = bN$. (The maximum phase lead occurs around $\omega = b\sqrt{N}$; the exact value depends on N and can be found in the Collection of Formulae.) See Figure 1.3.

1.5 **a.** In the Laplace domain, tracing backwards from the output y and setting all other inputs than r to zero, we obtain

$$Y = PC(R - Y) \quad \Leftrightarrow \quad Y = \frac{PC}{1 + PC} R$$

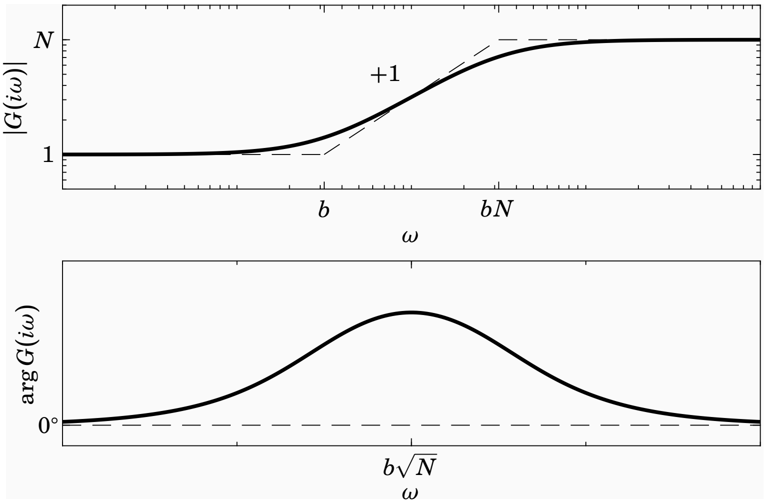

Figure 1.3 Bode diagram of $G(s) = \dfrac{N(s+b)}{s+bN}$

b.

$$V = -C(W_1 + PV) \quad \Leftrightarrow \quad V = \frac{-CW_1}{1+CP}W$$

1.6 a.

$$Y = (P_1 + P_2C_2)C_1(R_y - Y) \quad \Leftrightarrow \quad Y = \frac{(P_1 + P_2C_2)C_1}{1 + (P_1 + P_2C_2)C_1}R_y$$

b.

$$U_1 = -C_1(P_1U_1 + P_2C_2(U_1 - R_{u_1})) \quad \Leftrightarrow \quad U_1 = \frac{C_1P_2C_2}{1 + C_1(P_1 + P_2C_2)}R_{u_1}$$

1.7 a. The eigenvalues λ of the A matrix are given by

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{vmatrix} = \lambda^2 + 3\lambda + 2 = 0 \quad \Rightarrow \quad \lambda_1 = -1, \lambda_2 = -2$$

The system is stable since both eigenvalues are in the left half-plane.

b. Since the A matrix is triangular, we can immediately read out the eigenvalues as the diagonal elements: $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = 3$. The system is unstable since one eigenvalue is in the right half-plane.

c. The characteristic polynomial (the denominator of the transfer function) has positive coefficients, which is both a sufficient and a necessary condition for the stability of second-order systems.

d. A necessary condition for stability is that all coefficients of the characteristic polynomial have the same sign, which is not the case here.

1.8 a. The closed loop is given by

$$\frac{G_0K}{1+G_0K} = \frac{10K}{(s+1)(s+2)(s+3)+10K}$$

with the characteristic polynomial $s^3 + 6s^2 + 11s + 6 + 10K$. A third-order polynomial $s^3 + a_1s^2 + a_2s + a_3$ has LHP roots iff all coefficients are positive and $a_1a_2 > a_3$. We get $6 \cdot 11 > 6 + 10K$ and $6 + 10K > 0$. In total,

$$0 < K < 6$$

b. From the Bode diagram we can read that the Nyquist curve crosses the negative real axis ($\arg G_0 = -180°$) at around $\omega_0 \approx 3.2$ rad/s and that $|G(i\omega_0)| \approx 0.15$. The gain margin is hence $A_m = \frac{1}{0.15} \approx 6$. K must be smaller than this value to guarantee the Nyquist curve does not encircle -1.

1.9 **a.** The block diagram immediately gives

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ \frac{1}{s(s+1)} & \frac{1}{s(s+2)} \end{bmatrix}$$

b. Introducing one state variable per subsystem output, one obtains

$$\begin{aligned} \dot{x}_1 &= -x_1 + u_1 \\ \dot{x}_2 &= -2x_2 + u_2 \\ \dot{x}_3 &= x_1 + x_2 \\ y_1 &= x_1 + x_2 \\ y_2 &= x_3 \end{aligned}$$

1.10 **a.** The system matrices are

$$A = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

b. The transfer matrix is given by

$$G(s) = C(sI - A)^{-1}B + D = \begin{bmatrix} \frac{2s+6}{s^2+5s+6} & \frac{2s+6}{s^2+5s+6} \\ \frac{s^2+7s+14}{s^2+5s+6} & \frac{4}{s^2+5s+6} \\ \frac{2}{s+3} & \frac{s+7}{s+3} \end{bmatrix}$$

1.11 **a.** Laplace transformation of the differential equation gives

$$Y(s) = \frac{b_{11}s + b_{12}}{s^2 + a_1 s + a_2} U_1(s) + \frac{b_{21}s + b_{22}}{s^2 + a_1 s + a_2} U_2(s)$$

The transfer matrix becomes

$$\begin{bmatrix} \frac{b_{11}s + b_{12}}{s^2 + a_1 s + a_2} & \frac{b_{21}s + b_{22}}{s^2 + a_1 s + a_2} \end{bmatrix}$$

b. We can use for instance the observable canonical form and let each input generate one column in the B matrix. The state-space realization then becomes

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -a_1 & 1 \\ -a_2 & 0 \end{bmatrix} x(t) + \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{aligned}$$