Lecture 14 – Outline

1. Model reduction by balanced truncation

Model reduction

- Mathematical modeling can lead to dynamical models of very high order
- Controller synthesis using the Q-parameterization can lead to very high order controllers

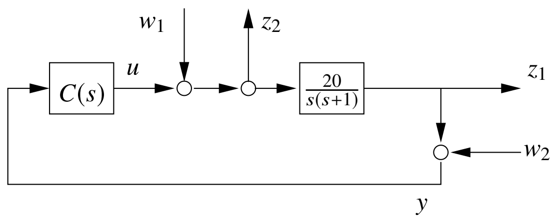
Need for systematic way to reduce the model order

In general terms we would like to achieve

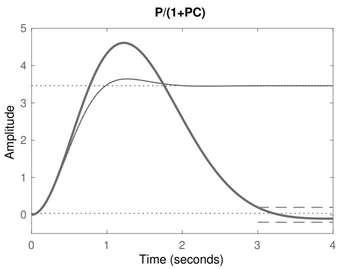
$$G_r(s) \approx G(s)$$

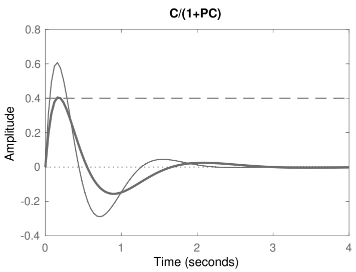
where $G_r(s)$ has (much) lower order than $G(s)$

Example – DC-motor



In Lecture 13 we minimized $\int_{-\infty}^{\infty} |G_{zw}(i\omega)|^2 d\omega$ subject to step response bounds on $G_{z_1w_1}$ and $G_{z_2w_2}$:





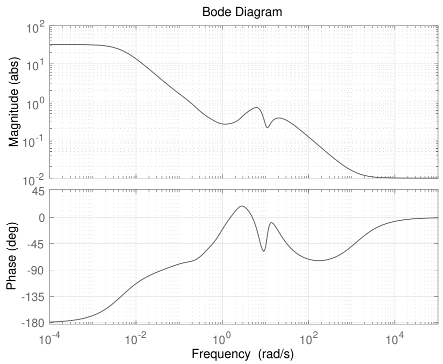
Example – DC-motor

Recall that

$$C(s) = \left[I + Q(s)P_{yu}(s)\right]^{-1} Q(s), \text{ with } Q(s) = \sum_{k=0}^{N} Q_k \phi_k(s).$$

Controller order grows with the number of basis functions.

Optimized controller for DC-motor has order 14. Is that really needed?



Controllability and observability Gramians

For a stable system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

the controllability Gramian $W_c = \int_0^\infty e^{At} BB^T e^{A^T t} dt$ is found by solving

$$AW_c + W_c A^T + BB^T = 0$$

and the observability Gramian $W_o = \int_0^\infty e^{A^T t} C^T C e^{At} dt$ is found by solving

$$A^T W_o + W_o A + C^T C = 0$$

Idea for model reduction: Remove states that are both poorly controllable and poorly observable.

Hankel singular values

The **Hankel singular values** are defined as the square roots of the eigenvalues of $W_c W_o$:

$$\sigma_i = \sqrt{\lambda_i(W_c W_o)}$$

They measure the "energy" of each mode in the system and are usually ordered such that

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$$

Matlab: `sigmas = hsvd(sys)`
(Unstable modes are assigned the value ∞)

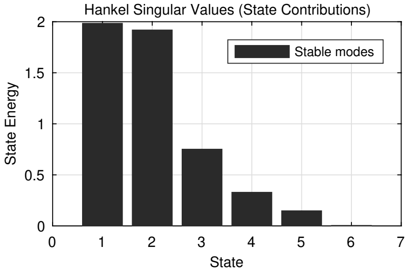
Example

System:

$$G(s) = \frac{1 - s}{s^6 + 3s^5 + 5s^4 + 7s^3 + 5s^2 + 3s + 1}$$

Hankel singular values (independent of realization):

$$\sigma = \begin{bmatrix} 1.984 & 1.918 & 0.751 & 0.329 & 0.148 & 0.004 \end{bmatrix}$$



Balanced realizations

Given a stable system (A, B, C, D) with Gramians W_c and W_o, the variable transformation $\hat{x} = Tx$ gives the new state-space matrices $\hat{A} = TAT^{-1}$, $\hat{B} = TB$, $\hat{C} = CT^{-1}$, $\hat{D} = D$ and the new Gramians

$$\hat{W}_c = TW_cT^T$$
$$\hat{W}_o = T^{-T} W_o T^{-1}$$

A particular choice of T gives $\hat{W}_c = \hat{W}_o = \Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}$

The corresponding realization $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is called a **balanced realization**.

Computing the balancing state transformation

(Not done by hand)

Compute the Cholesky decompositions

$$W_c = WW^T, \quad W_o = ZZ^T$$

and the singular value decomposition

$$W^T Z = U\Sigma V^T$$

The balancing transformation is then given by

$$T = \Sigma^{-\frac{1}{2}} V^T Z^T, \quad T^{-1} = WU\Sigma^{-\frac{1}{2}}$$

Matlab: `[sysb,sigmas,T] = balreal(sys)`

Hankel singular values and truncation

Notice that

$$\begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{bmatrix} = \underbrace{(TW_cT^T)}_{\Sigma}\underbrace{(T^{-T}W_oT^{-1})}_{\Sigma} = TW_cW_oT^{-1}$$

so the Hankel singular values are independent of the coordinate system.

A small Hankel singular value σ_i corresponds to a state that is both weakly controllable and weakly observable. Hence, it can be truncated without much effect on the input-output behavior.

Model reduction by balanced truncation

Consider a balanced realization

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \qquad \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + Du$$

with the lower part of the Gramian being $\Sigma_2 = \mathrm{diag}(\sigma_{r+1}, \ldots, \sigma_n)$.
Two ways to do the reduction:

1. Simply remove $\hat{x}_2$ and keep (A_{11}, B_1, C_1, D).
2. (Default:) Set $\dot{\hat{x}}_2 = 0$. Gives the reduced system

$$\begin{cases} \dot{\hat{x}}_1 = (A_{11} - A_{12}A_{22}^{-1}A_{21})\hat{x}_1 + (B_1 - A_{12}A_{22}^{-1}B_2)u \\ y_r = (C_1 - C_2A_{22}^{-1}A_{21})\hat{x}_1 + (D - C_2A_{22}^{-1}B_2)u \end{cases}$$

Error bounds for balanced truncation

One way to measure the approximation error between the original system $G(s)$ and the reduced system $G_r(s)$ is

$$\|G - G_r\|_\infty = \max_\omega |G(i\omega) - G_r(i\omega)| = \sup_u \frac{\|y - y_r\|_2}{\|u\|_2}$$

For either of the truncation methods above, it holds that

$$\sigma_{r+1} \ \le\ \|G - G_r\|_\infty \ \le\ 2(\sigma_{r+1} + \cdots + \sigma_n)$$

Example

System:

$$G(s) = \frac{1 - s}{s^6 + 3s^5 + 5s^4 + 7s^3 + 5s^2 + 3s + 1}$$

Keeping $r = 3$ states gives the reduced system (default method):

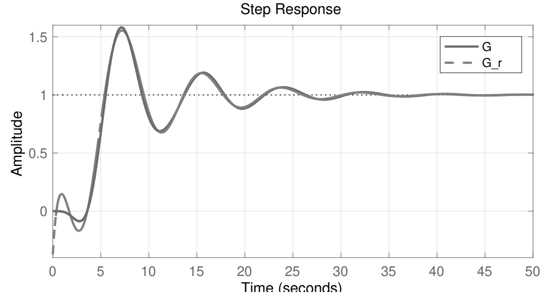
$$G_r(s) = \frac{0.3717s^3 - 0.9682s^2 + 1.14s - 0.5185}{s^3 + 1.136s^2 + 0.825s + 0.5185}$$

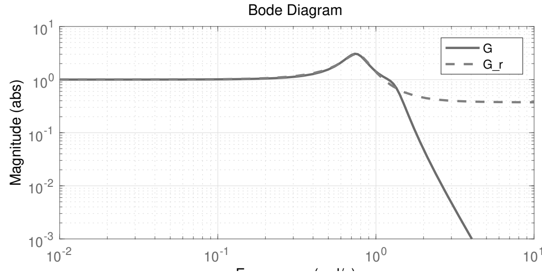
Error bounds: $0.329 \leq \|G - G_r\|_\infty \leq 0.963$

Actual error: $\|G - G_r\|_\infty = 0.573$

Matlab: `[Gbal,sigmas]=balreal(G); Gred=modred(Gbal,4:6)`

Example





Handling unstable systems

Before model reduction, decompose the system into its stable and nonstable parts:

$$G(s) = G_s(s) + G_{ns}(s)$$

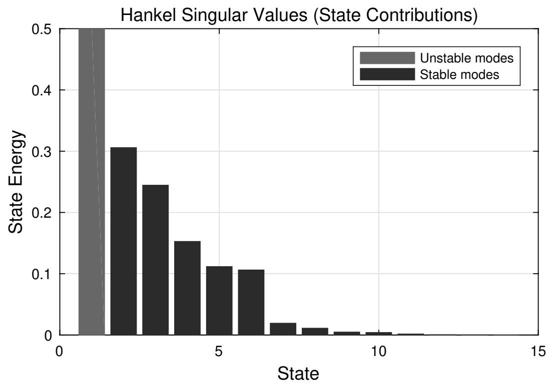
Perform the reduction only on $G_s(s)$; then add $G_{ns}(s)$ again

(Performed automatically by Matlab's `balreal` and `balred`)

Example – DC-motor

Computing the 14 Hankel singular values gives

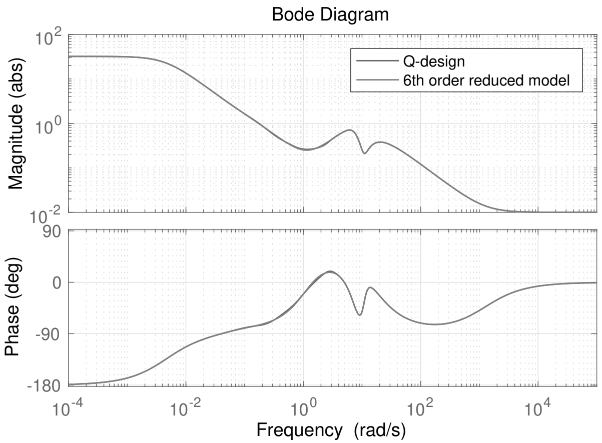
$$\begin{bmatrix} \infty & 0.306 & 0.244 & 0.153 & 0.115 & 0.106 & 0.019 & 0.011 & \ldots \end{bmatrix}$$



The unstable mode is excluded from the reduction.

Example – DC-motor

Straight truncation gives reduced controller with 6 states:

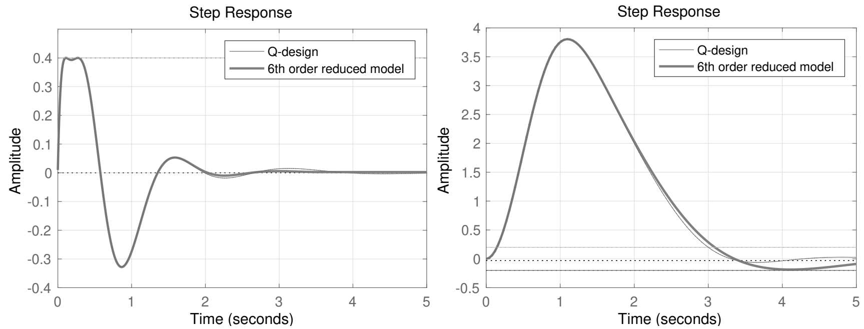


Matlab:

```
ctrl_red=balred(ctrl_opt,6,'StateElimMethod','Truncate')
```


Example – DC-motor

Are the design specifications still satisfied?



Almost. . .

Summary

- Low-order controllers are preferred from an implementation point of view (execution time, memory usage)
- Balanced realizations reveal the less important states
- Model reduction by balanced trunction has good theoretical error bounds
- Many possible extensions, e.g.
  - optimal model reduction (non-convex problem)
  - frequency weighting
  - reduction of unstable systems