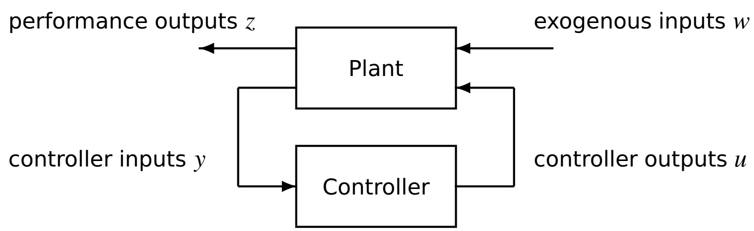
Lecture 13 – Outline

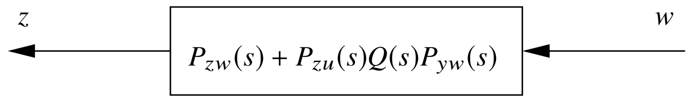
1. Examples
2. Introduction to convex optimization
3. Controller optimization using Youla parameterization
4. Examples revisited

Parts of this lecture is based on material from Boyd, Vandenberghe and coauthors. See also lecture notes and links on course homepage.

General idea for Lectures 12–14



The choice of controller corresponds to designing a transfer matrix $Q(s)$, to get desirable properties of the following map from w to z:



Once $Q(s)$ has been designed, the corresponding controller can be found.

Example 1 (Doyle–Stein, 1979)

Given the process

$$\dot{x} = \begin{pmatrix} -4 & -3 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u + \begin{pmatrix} -61 \\ 35 \end{pmatrix} w_1$$
$$y = \begin{pmatrix} 1 & 2 \end{pmatrix} x + w_2$$

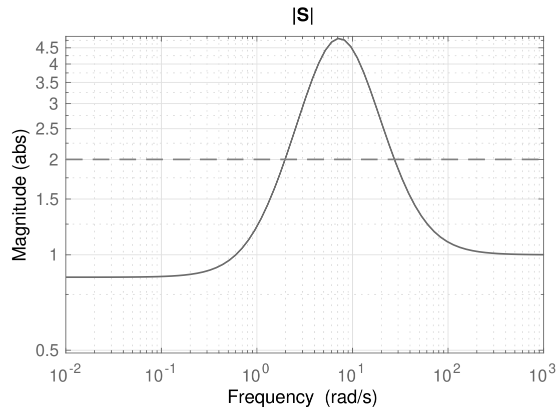
where w_1 and w_2 are independent unit-intensity white noise processes, find a controller that minimizes

$$J = \mathrm{E}\left\{ 80\, x^T \begin{pmatrix} 1 & \sqrt{35} \\ \sqrt{35} & 35 \end{pmatrix} x + u^2 \right\}$$

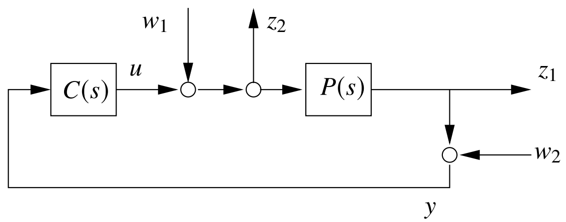
while satisfying the robustness constraint $M_s \leq 2$

Example 1 (Doyle–Stein, 1979)

LQG design gives a controller that does not satisfy the constraint on $|S|$ (see Lecture 11):



Example 2 – DC-motor



Assume we want to optimize the closed-loop transfer matrix from $(w_1, w_2)^T$ to $(z_1, z_2)^T$,

$$G_{zw}(s) = \begin{bmatrix} \frac{P}{1-PC} & \frac{PC}{1-PC} \\ \frac{1}{1-PC} & \frac{C}{1-PC} \end{bmatrix}$$

when $P(s) = \frac{20}{s(s+1)}$.

Example 2 – DC-motor

It can be shown that minimizing

$$\int_{-\infty}^{\infty} |G_{zw}(i\omega)|^2 \, d\omega$$

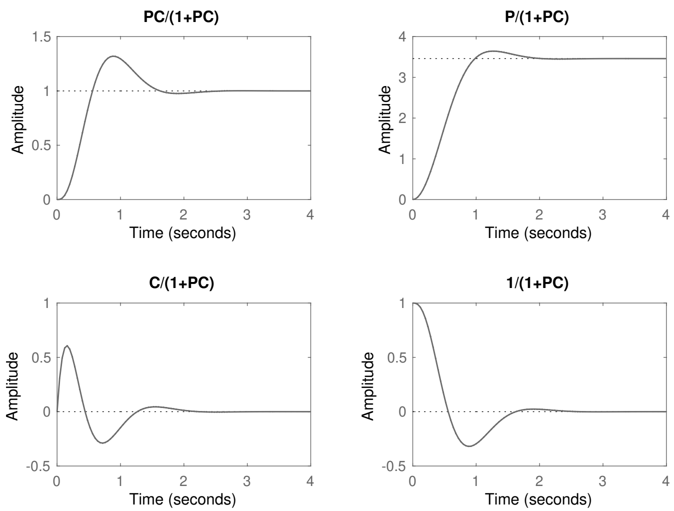
is equivalent to solving the LQG problem with (see Lecture 11)

$$A = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \ B = G = \begin{pmatrix} 20 \\ 0 \end{pmatrix}, \ C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$Q_1 = C^T C, \ Q_2 = R_1 = R_2 = 1$$

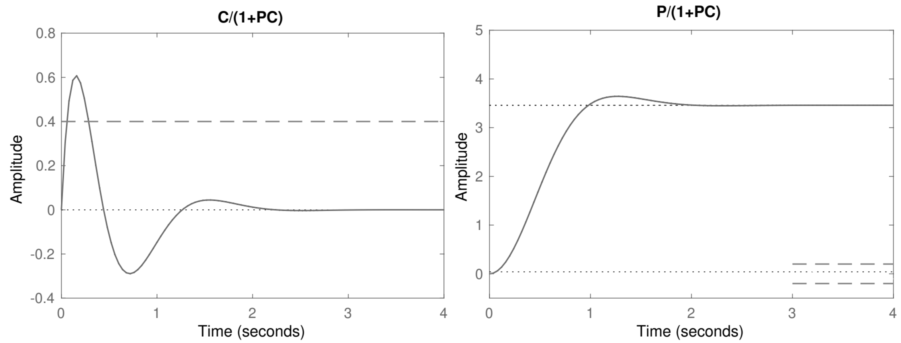
Example 2 – DC-motor

"Gang of four" step responses:



Example 2 – DC-motor

Suppose we want to add some time-domain constraints:



- Control signal $|u| \leq 0.4$ for unit output disturbance (or setpoint change)
- Output signal $|y| \leq 0.2$ for $t \geq 3$ for unit load disturbance

Lecture 13 – Outline

Convex optimization

Convex optimization = minimization of convex function over convex set

- Also known as convex programming
- Key property: Any local minimum must also be a global minimum
- Convex problems **can** be solved, and efficient solvers are available
  - By contrast, most **nonconvex** problems **cannot** be solved
- Many engineering design problems can be formulated as convex optimization problems

Mathematical formulation

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le b_i, \quad i = 1, \ldots, m \end{array}$$

- objective and constraint functions are convex:

$$f_i(\alpha x + \beta y) \le \alpha f_i(x) + \beta f_i(y)$$

  if $\alpha + \beta = 1$, $\alpha \ge 0$, $\beta \ge 0$

- includes least-squares problems and linear programs as special cases

Least squares

$$\text{minimize} \quad \|Ax - b\|_2^2$$

solving least-squares problems

- analytical solution: $x^\star = (A^T A)^{-1} A^T b$
- reliable and efficient algorithms and software
- computation time proportional to $n^2 k$ ($A \in \mathbf{R}^{k \times n}$); less if structured
- a mature technology

using least-squares

- least-squares problems are easy to recognize
- a few standard techniques increase flexibility ($e.g.$, including weights, adding regularization terms)

Linear program (LP)

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & a_i^T x \le b_i, \quad i = 1, \ldots, m
\end{array}
$$

solving linear programs

- no analytical formula for solution
- reliable and efficient algorithms and software
- computation time proportional to $n^2 m$ if $m \ge n$; less with structure
- a mature technology

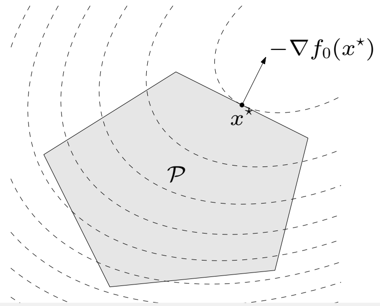
using linear programming

- not as easy to recognize as least-squares problems
- a few standard tricks used to convert problems into linear programs ($e.g.$, problems involving ℓ_1- or ℓ_∞-norms, piecewise-linear functions)

Quadratic program (QP)

$$
\begin{array}{ll}
\text{minimize} & (1/2)x^T P x + q^T x + r \\
\text{subject to} & Gx \preceq h \\
& Ax = b
\end{array}
$$

- $P \in \mathbf{S}_+^n$, so objective is convex quadratic
- minimize a convex quadratic function over a polyhedron



General convex program

solving convex optimization problems

- no analytical solution
- reliable and efficient algorithms
- computation time (roughly) proportional to $\max\{n^3, n^2 m, F\}$, where F is cost of evaluating f_i's and their first and second derivatives
- almost a technology

using convex optimization

- often difficult to recognize
- many tricks for transforming problems into convex form
- surprisingly many problems can be solved via convex optimization

Brief history of convex optimization

theory (convex analysis): ca1900–1970

algorithms

- 1947: simplex algorithm for linear programming (Dantzig)
- 1960s: early interior-point methods (Fiacco & McCormick, Dikin, . . .)
- 1970s: ellipsoid method and other subgradient methods
- 1980s: polynomial-time interior-point methods for linear programming (Karmarkar 1984)
- late 1980s–now: polynomial-time interior-point methods for nonlinear convex optimization (Nesterov & Nemirovski 1994)

applications

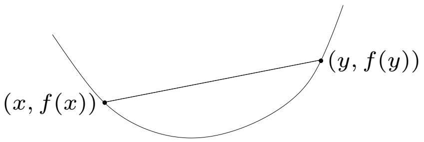
- before 1990: mostly in operations research; few in engineering
- since 1990: many new applications in engineering (control, signal processing, communications, circuit design, . . .); new problem classes (semidefinite and second-order cone programming, robust optimization)

Definition of convex function

$f : \mathbf{R}^n \to \mathbf{R}$ is convex if $\mathbf{dom}\, f$ is a convex set and

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$

for all $x, y \in \mathbf{dom}\, f$, $0 \leq \theta \leq 1$



- f is concave if $-f$ is convex
- f is strictly convex if $\mathbf{dom}\, f$ is convex and

$$f(\theta x + (1-\theta)y) < \theta f(x) + (1-\theta) f(y)$$

for $x, y \in \mathbf{dom}\, f$, $x \neq y$, $0 < \theta < 1$

Examples on R

convex:

- affine: $ax + b$ on $\mathbf{R}$, for any $a, b \in \mathbf{R}$
- exponential: e^{ax}, for any $a \in \mathbf{R}$
- powers: x^α on $\mathbf{R}_{++}$, for $\alpha \geq 1$ or $\alpha \leq 0$
- powers of absolute value: $|x|^p$ on $\mathbf{R}$, for $p \geq 1$
- negative entropy: $x \log x$ on $\mathbf{R}_{++}$

concave:

- affine: $ax + b$ on $\mathbf{R}$, for any $a, b \in \mathbf{R}$
- powers: x^α on $\mathbf{R}_{++}$, for $0 \leq \alpha \leq 1$
- logarithm: $\log x$ on $\mathbf{R}_{++}$

Examples on $\mathbf{R}^n$ and $\mathbf{R}^{m \times n}$

affine functions are convex and concave; all norms are convex

examples on $\mathbf{R}^n$

- affine function $f(x) = a^T x + b$
- norms: $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $p \geq 1$; $\|x\|_\infty = \max_k |x_k|$

examples on $\mathbf{R}^{m \times n}$ ($m \times n$ matrices)

- affine function

$$f(X) = \mathbf{tr}(A^T X) + b = \sum_{i=1}^m \sum_{j=1}^n A_{ij} X_{ij} + b$$

- spectral (maximum singular value) norm

$$f(X) = \|X\|_2 = \sigma_{\max}(X) = (\lambda_{\max}(X^T X))^{1/2}$$

Solving convex programs

- Specialized methods for different subtypes of convex programs
- Medium-scale problems (thousands of variables and constraints) can be solved using standard interior point methods
  - Relax the constraints using barrier functions
  - Use Newton's method in each iteration while gradually sharpening the barriers
- Large-scale problems (millions or billions of variables and constraints) require special methods and special software

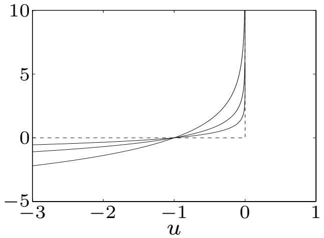
Barrier method for constrained minimization

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0 \quad 1 = 1, \dots, m \\ & Ax = b \end{array}$$

approximation via logarithmic barrier

$$\begin{array}{ll} \text{minimize} & f_0(x) - (1/t)\sum_{i=1}^{m} \log(-f_i(x)) \\ \text{subject to} & Ax = b \end{array}$$

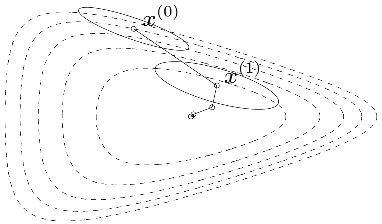
- an equality constrained problem
- for $t > 0$, $-(1/t)\log(-u)$ is a smooth approximation of I_-
- approximation improves as $t \to \infty$

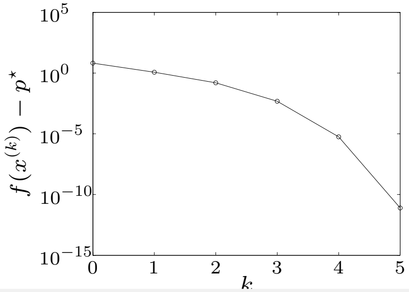


Newton's method

given a starting point $x \in \mathbf{dom}\, f$, tolerance $\epsilon > 0$.
repeat

1. *Compute the Newton step and decrement.*
   $\Delta x_{\text{nt}} := -\nabla^2 f(x)^{-1}\nabla f(x); \quad \lambda^2 := \nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x).$
2. *Stopping criterion.* **quit** if $\lambda^2/2 \le \epsilon$.
3. *Line search.* Choose step size t by backtracking line search.
4. *Update.* $x := x + t\Delta x_{\text{nt}}$.





Software for convex optimization

- CVX – Matlab toolbox for disciplined convex programming, developed at Stanford by Stephen Boyd and co-workers
  - Easily integrated with Python, Julia
  - CVXGEN – C code generation
- YALMIP – Matlab toolbox for convex and nonconvex optimization problems
- Solvers (plugins):
  - SeDuMi – software for optimization over symmetric cones
  - SDPT3 – software for semidefinite programming
  - Mosek – commercial optimization software
  - Gurobi – commercial optimization software

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Scheme for numerical optimization of Q

Given some fixed set of basis function $\phi_0(s), \dots, \phi_N(s)$, we will search numerically for matrices $Q_0, \dots, Q_N$ such that the closed-loop matrix $G_{zw}(s)$ satisfies given specifications when

$$G_{zw}(s) = P_{zw}(s) + P_{zu}(s)Q(s)P_{yw}(s) \qquad \text{and } Q(s) = \sum_{k=0}^{N} Q_k \phi_k(s)$$

It is possible to choose the sequence $\phi_0(s), \phi_1(s), \phi_2(s), \dots$ such that every stable Q can be approximated arbitrarily well. In principle, every convex control design problem can be solved this way.

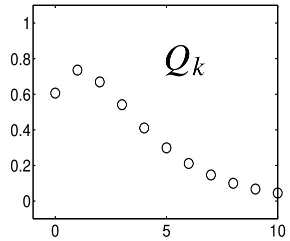
Choice of basis functions

Many possibilities. Common choices:

- Simplified Laguerre basis polynomials,

$$\phi_k(s) = \frac{1}{(s/a+1)^k}$$

  where a should be wisely selected
  (rule of thumb: close to bandwidth of closed-loop system)
- Pulse response parameterization (discrete time approximation)



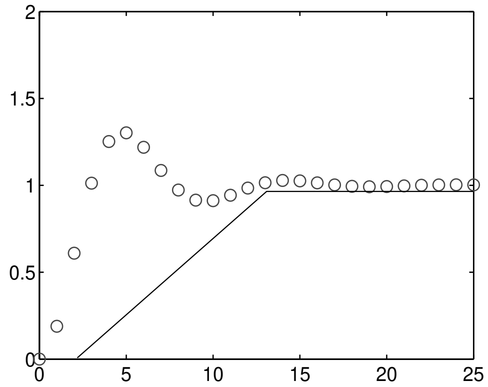
Specifications that lead to convex constraints

- Stability of the closed-loop system
- Upper and lower bounds on step response from w_i to z_j at time t_i
- Upper bound on Bode amplitude from w_i to z_j at frequency ω_i
- Interval bound on Bode phase from w_i to z_j at frequency ω_i

The following constraints are however **nonconvex**:

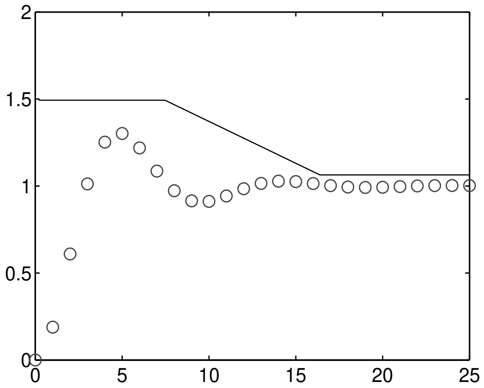
- Stability of the controller
- Lower bound on Bode amplitude from w_i to z_j at frequency ω_i

Lower bound on step response



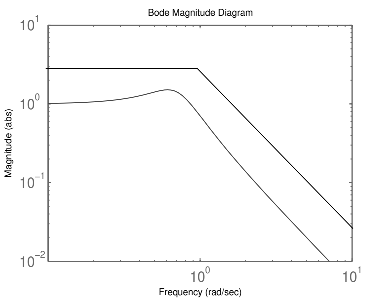
The step response depends linearly on Q_k, so every time t_k with a lower bound gives a linear constraint.

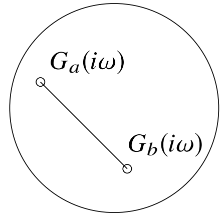
Upper bound on step response



Every time t_k with an upper bound also gives a linear constraint.

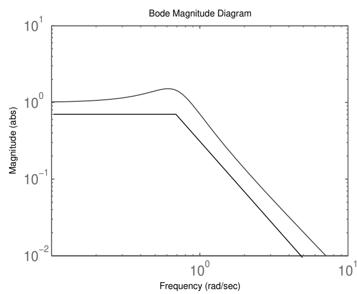
Upper bound on Bode amplitude

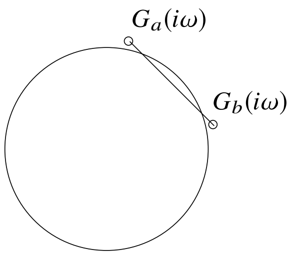




An amplitude bound $|G(i\omega_i)| < c$ is a quadratic constraint.

Lower bound on Bode amplitude





An lower bound $|G(i\omega_i)|$ is a **nonconvex** quadratic constraint. This should be avoided in optimization.

Synthesis by convex optimization

Quite general control synthesis problems can be stated as convex optimization problems in the variable $Q(s)$. The problem could have a quadratic objective, with linear/quadratic constraints, e.g.:

$$\min \quad \int_{-\infty}^{\infty} \Big|P_{zw}(i\omega) + P_{zu}(i\omega) \overbrace{\sum_k Q_k \phi_k(i\omega)}^{Q(i\omega)} P_{yw}(i\omega)\Big|^2 d\omega \Big\} \text{ quadratic objective}$$

s.t.
- step response $w_i \longrightarrow z_j$ is smaller than f_{ijk} at time t_k } linear constraints
- step response $w_i \longrightarrow z_j$ is bigger than g_{ijk} at time t_k } linear constraints
- Bode magnitude $w_i \longrightarrow z_j$ is smaller than h_{ijk} at ω_k } quadratic constraints

Here $Q(s) = \sum_k Q_k \phi_k(s)$, where $\phi_1, \ldots, \phi_m$ are some fixed basis functions, and $Q_0, \ldots, Q_m$ are optimization variables.

Once $Q(s)$ has been determined, the controller is obtained as

$$C(s) = \left[I + Q(s)P_{yu}(s)\right]^{-1} Q(s)$$

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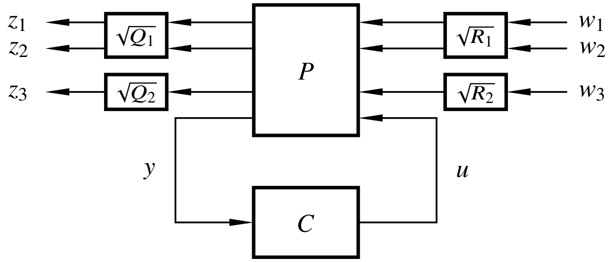
Example 1 (Doyle–Stein, 1979)

LQG problem reformulated as extended plant model:



Minimize

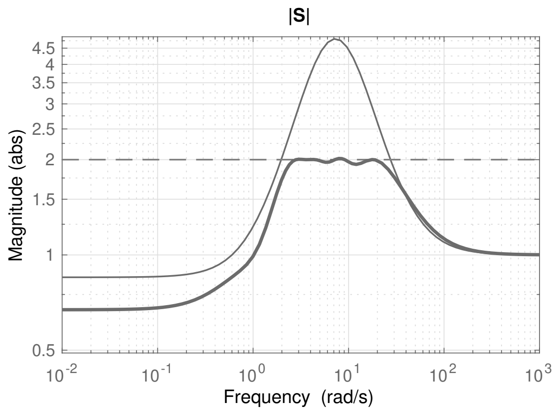
$$\int_{-\infty}^{\infty} |P_{zw}(i\omega) + P_{zu}(i\omega)\sum_k q_k \phi_k(i\omega) P_{yw}(i\omega)|^2 d\omega$$

with q_k scalar and

$$\phi_k(s) = \frac{1}{(s/a+1)^k}$$

Example 1 (Doyle–Stein, 1979)

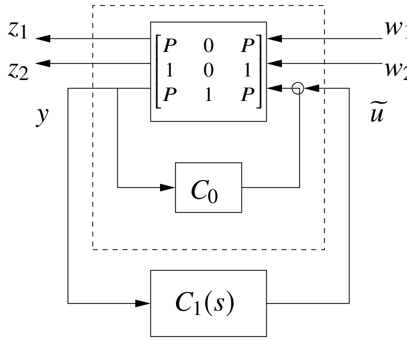
Green: Optimization-based design with constraint on $|S|$:

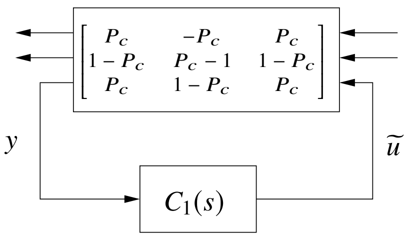


(Controller order: 12)

Example 2 – DC-servo

Introduce stabilizing controller C_0 and reformulate for optimization:

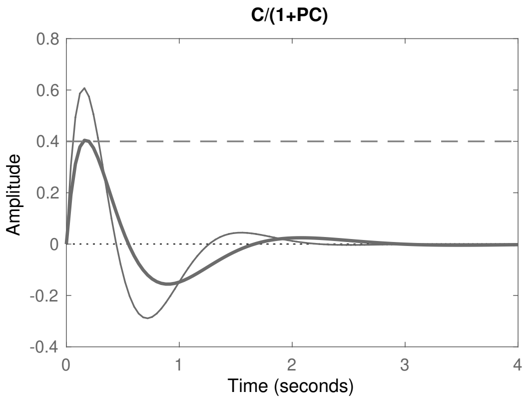




$$G_{zw}(s) = \begin{bmatrix} P_c & -P_c \\ 1-P_c & P_c - 1 \end{bmatrix} + \begin{bmatrix} P_c \\ 1-P_c \end{bmatrix} Q \begin{bmatrix} P_c & 1-P_c \end{bmatrix}$$

Example 2 – DC-servo

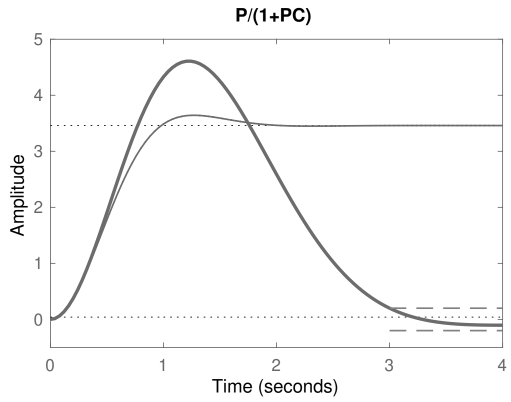
Green: Optimization with control signal limitation:



(Controller order: 14)

Example 2 – DC-servo

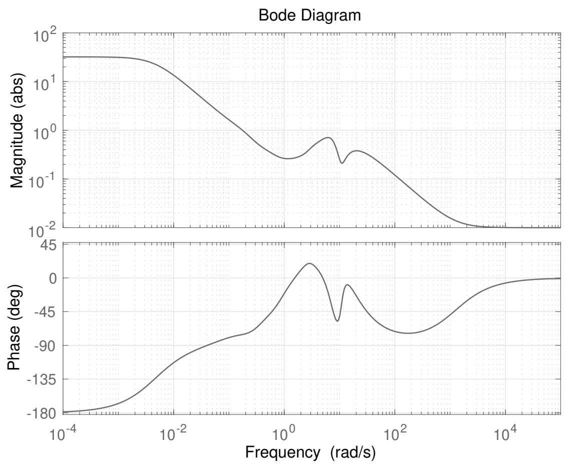
Green: Also adding the limit on y, $3 \leq t \leq 4$:



(Controller order: 14)

Example 2 – DC-servo

Final controller:



Is it any good? With optimization, you get what you ask for!

Lecture 13 – summary

- There are efficient algorithms for solving convex programs
  - Local optimum ⇔ global optimum
- The Youla parameterization allows us to use these algorithms for control synthesis
- Resulting controllers typically have high order. Order reduction will be studied in the next lecture.

Further reading: Stephen Boyd's books on convex optimization are available online:

`http://stanford.edu/~boyd/books.html`