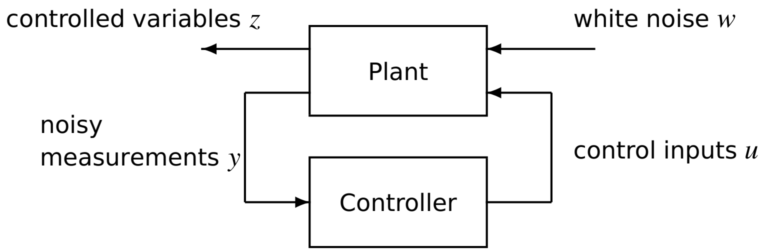
Lecture 10 – Outline

1. Observer-based feedback
2. The Kalman filter

Goal: Linear-quadratic-Gaussian control (LQG)

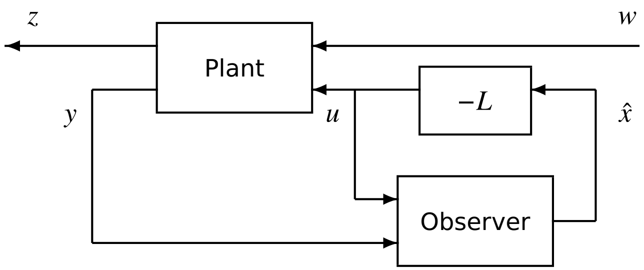


For a linear plant, let w be white noise of intensity R. Find a controller that minimizes the performance index

$$\mathrm{E}\,|z|^2 = \mathrm{E}\left\{x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u\right\}$$

Previous lecture: State feedback solution ($y = x,$ no noise)

Output feedback using an observer



Plant:
$$\begin{cases} \frac{dx(t)}{dt} = Ax(t) + Bu(t) + w_1(t) & \text{(process disturbances)} \\ y(t) = Cx(t) \qquad\qquad + w_2(t) & \text{(measurement noise)} \end{cases}$$

Controller:
$$\begin{cases} \frac{d\hat{x}(t)}{dt} = A\hat{x}(t) + Bu(t) + K[y(t) - C\hat{x}(t)] \\ u(t) = -L\hat{x}(t) \end{cases}$$

Closed-loop dynamics

Eliminate u and y:

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) - BL\hat{x}(t) + w_1(t) \\ \frac{d\hat{x}(t)}{dt} &= A\hat{x}(t) - BL\hat{x}(t) + K[Cx(t) - C\hat{x}(t)] + Kw_2(t) \end{aligned}$$

Introduce the observer error $\tilde{x} = x - \hat{x}$

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix} = \begin{bmatrix} A - BL & BL \\ 0 & A - KC \end{bmatrix}\begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix} + \begin{bmatrix} w_1(t) \\ w_1(t) - Kw_2(t) \end{bmatrix}$$

Two kinds of closed-loop poles:

Control poles: $0 = \det(sI - A + BL)$

Observer poles: $0 = \det(sI - A + KC)$

L and K can be designed independent from each other

State observer

Dual goals:

- Estimate state variables that cannot be directly measured
- Filter out measurement noise

What is the optimal balance between speed of estimation and noise reduction?

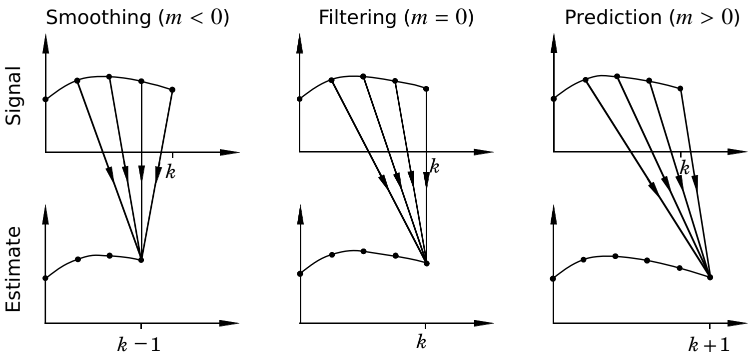
Rudolf E. Kálmán, 1930–2016

Recipient of the 2008 Charles Stark Draper Prize from the US National Academy of Engineering "for the devlopment and dissemination of the optimal digital technique (known as the Kalman Filter) that is pervasively used to control a vast array of consumer, health, commercial and defense products."

Optimal filtering and prediction

- Wiener (1949): Stationary input-output formulation
- Kalman (1960): Time-varying state-space formulation (discrete time) ["A new approach to linear filtering and prediction problems", *Transactions of ASME–Journal of Basic Engineering*, **82**]

General problem: Estimate $x(k+m)$ given $\{y(i),\ u(i) \mid i \leq k\}$



Examples

Smoothing	To estimate the Wednesday temperature based on measurements from Tuesday, Wednesday and Thursday
Filtering	To estimate the Wednesday temperature based on measurements from Monday, Tuesday and Wednesday
Prediction	To predict the Wednesday temperature based on measurements from Sunday, Monday and Tuesday

The optimal observer problem

The observer error dynamics are given by

$$\frac{d\tilde{x}}{dt} = (A - KC)\tilde{x} + \begin{pmatrix} I & -K \end{pmatrix} w$$

The disturbance $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ is assumed white with intensity

$$\Phi_w(\omega) = \begin{pmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{pmatrix} > 0$$

Optimization problem: Assuming that the system is detectable (any unstable modes are observable), find the gain K that minimizes the stationary observer error variance

$$P = \mathrm{E}\,\tilde{x}\,\tilde{x}^T$$

Finding the optimal observer gain

The stationary observer error variance P is given by the Lyapunov equation

$$(A - KC)P + P(A - KC)^T + \begin{pmatrix} I & -K \end{pmatrix} \begin{pmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{pmatrix} \begin{pmatrix} I \\ -K^T \end{pmatrix} = 0$$

Completing the square,

$$AP + PA^T + R_1 + (K - (PC^T + R_{12})R_2^{-1})R_2(K - (PC^T + R_{12})R_2^{-1})^T$$
$$-(PC^T + R_{12})R_2^{-1}(PC^T + R_{12})^T = 0$$

we find that the minimium variance is attained for

$$K = (PC^T + R_{12})R_2^{-1}$$

What remains is an algebraic Riccati equation,

$$AP + PA^T + R_1 - (PC^T + R_{12})R_2^{-1}(PC^T + R_{12})^T = 0$$

The Kalman filter

Given a detectable linear plant disturbed by white noise,

$$\begin{cases} \dot{x} = Ax + Bu + w_1 \\ y = Cx + w_2 \end{cases} \qquad \Phi_w = \begin{pmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{pmatrix} > 0$$

the optimal observer is given by

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + K(y - C\hat{x})$$

where K is given by

$$K = (PC^T + R_{12})R_2^{-1}$$

where $P = \mathrm{E}(x - \hat{x})(x - \hat{x})^T > 0$ is the solution to

$$AP + PA^T + R_1 - (PC^T + R_{12})R_2^{-1}(PC^T + R_{12})^T = 0$$

Remarks

The optimal observer gain does not depend on what state(s) we are interested in. The Kalman filter produces the optimal estimate of **all states** at the same time.

The optimal observer gain K is static since we are solving a steady-state problem.

(The Kalman filter can also be derived for finite-horizon problems and problems with time-varying system matrices. We then obtain a Riccati differential equation for $P(t)$ and a time-varying filter gain $K(t)$)

Duality between LQ control and Kalman filtering

LQ control	Kalman filter
A	A^T
B	C^T
Q_1	R_1
Q_2	R_2
Q_{12}	R_{12}
S	P
L	K^T

Matlab:

```
S = care(A,B,Q1,Q2,Q12)
P = care(A',C',R1,R2,R12)
```


Alternative state-space models

A common alternative state-space description is

$$\dot{x} = Ax + Bu + Gv_1 \qquad \Phi_v = \begin{pmatrix} R_{v_1} & R_{v_{12}} \\ R_{v_{12}}^T & R_{v_2} \end{pmatrix}$$
$$y = Cx + v_2$$

This is equivalent to

$$\dot{x} = Ax + Bu + w_1 \qquad \Phi_w = \begin{pmatrix} GR_{v_1}G^T & GR_{v_{12}} \\ R_{v_{12}}^T G^T & R_{v_2} \end{pmatrix}$$
$$y = Cx + w_2$$

(See `lqe` and `kalman` below for even more variants)

Kalman filter in Matlab (1a)

```
lqe  Kalman estimator design for continuous-time systems.

   Given the system
       .
       x = Ax + Bu + Gw            {State equation}
       y = Cx + Du + v             {Measurements}

   with unbiased process noise w and measurement noise v with
   covariances

       E{ww'} = Q,    E{vv'} = R,    E{wv'} = N ,

   [L,P,E] = lqe(A,G,C,Q,R,N)  returns the observer gain matrix L
   such that the stationary Kalman filter
       .
       x_e = Ax_e + Bu + L(y - Cx_e - Du)

   produces an optimal state estimate x_e of x using the sensor
   measurements y.  The resulting Kalman estimator can be formed
   with ESTIM.
```


Kalman filter in Matlab (1b)

```
 estim  Form estimator given estimator gain.

   EST = estim(SYS,L) produces an estimator EST with gain L for
   the outputs and states of the state-space model SYS, assuming
   all inputs of SYS are stochastic and all outputs are measured.
   For continuous-time systems
             .
      SYS:   x = Ax + Bw ,   y = Cx + Dw   (with w stochastic),

   the estimator equations are
       .
       x_e   = [A-LC] x_e + Ly

     | y_e | = | C | x_e
     | x_e |   | I |

   and the outputs x_e(t) and y_e(t) of EST are estimates of x(t)
   and y(t)=Cx(t).
```

Kalman filter in Matlab (2)

```
kalman  Kalman state estimator.

   [KEST,L,P] = kalman(SYS,QN,RN,NN) designs a Kalman estimator KEST for
   the continuous- or discrete-time plant SYS. For continuous-time plants
      .
      x = Ax + Bu + Gw            {State equation}
      y = Cx + Du + Hw + v        {Measurements}

   with known inputs u, process disturbances w, and measurement noise v,
   KEST uses [u(t);y(t)] to generate optimal estimates y_e(t),x_e(t) of
   y(t),x(t) by:
       .
      x_e  = Ax_e + Bu + L (y - Cx_e - Du)

      |y_e| = | C | x_e + | D | u
      |x_e|   | I |       | 0 |

   kalman takes the state-space model SYS=SS(A,[B G],C,[D H]) and the
   covariance matrices:

      QN = E{ww'},     RN = E{vv'},     NN = E{wv'}.
```


Example 1 – Kalman filter for an integrator

$$\dot{x}(t) = w_1(t) \qquad \Phi_w = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}$$
$$y(t) = x(t) + w_2(t)$$

Kalman filter:

$$\frac{d\hat{x}}{dt} = A\hat{x}(t) + Bu(t) + K[y(t) - C\hat{x}(t)]$$

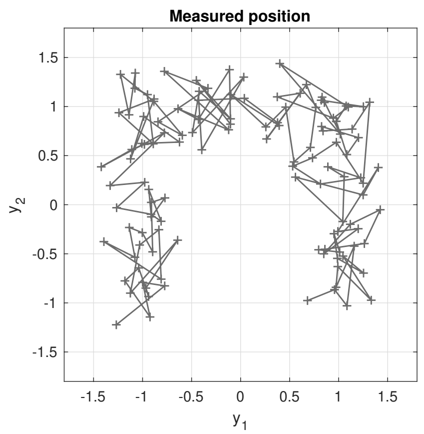
Riccati equation: $0 = R_1 - P^2/R_2 \quad \Rightarrow \quad P = \sqrt{R_1 R_2}$

Filter gain: $K = P/R_2 = \sqrt{R_1/R_2}$

Interpretation?

Example 2 – Tracking of a moving object

Position readings $y = (y_1, y_2)^T$ with measurement noise:



Would like to estimate the true position (p_1, p_2)

Example 2 – Tracking of a moving object

Dynamic model: Two double integrators driven by noise, $\ddot{p}_i = w_{1i}$

State vector: $x = \begin{pmatrix} p_1 & \dot{p}_1 & p_2 & \dot{p}_2 \end{pmatrix}^T$

State-space model:

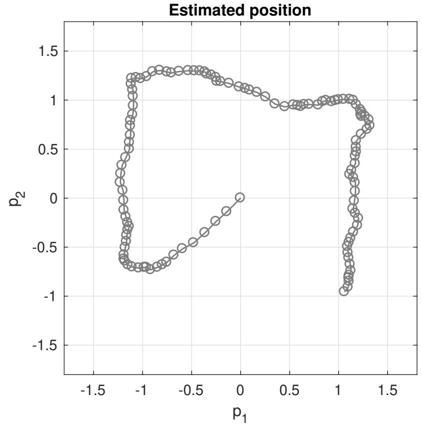
$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} w_1$$

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} x + w_2$$

Fix $R_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and design Kalman filters for different R_2

Example 2 – Tracking of a moving object

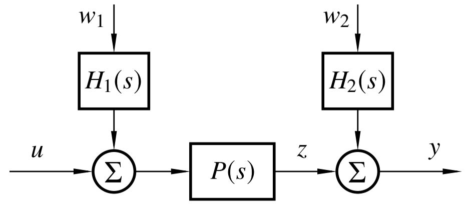
Simulation of Kalman filter with initial condition $\hat{x} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}^T$



Larger R_2 gives better noise rejection but slower tracking

Noise shaping

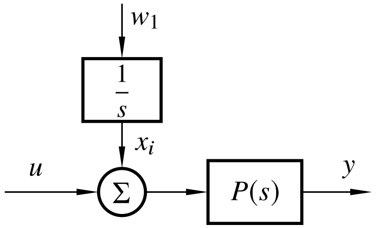
The Kalman filter can be tuned in the frequency domain by extending the plant model with filters that shape the process disturbance and measurement noise spectra:



- Process disturbance frequencies are modeled via H_1
  - Kalman filter gets higher gain where $|H_1(i\omega)|$ is large
- Measurement disturbance frequencies are modeled via H_2
  - Kalman filter gets smaller gain where $|H_2(i\omega)|$ is large

Integral action via noise shaping

Extend the plant model with an integral disturbance acting on the process input. (Extra state is observable but not controllable.)



The resulting Kalman filter (and hence also the observer-based controller) will include an integrator. Extended feedback law:

$$u(t) = -L\hat{x}(t) - \hat{x}_i(t)$$

Lecture 10 – summary

- Observer-based feedback
- The Kalman filter – an optimal observer
- Noise shaping

Next lecture: LQG:

- LQG by separation (LQ regulator + Kalman filter)
- Robustness of LQG?
- How to choose the design weights Q and R?
- How to handle reference signals?
- Examples