Lecture 8 – Outline

1. Transfer functions for MIMO systems
2. Limitations due to RHP zeros
3. Decentralized control
4. Decoupling

Typical process control system

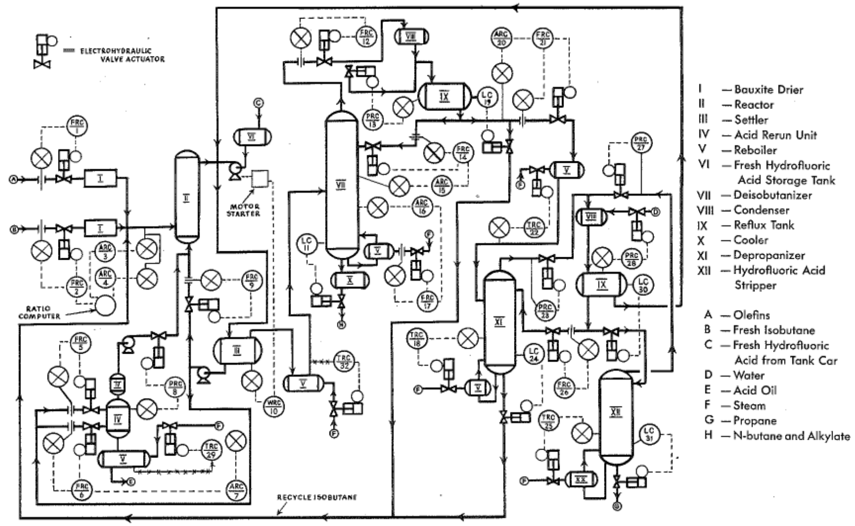
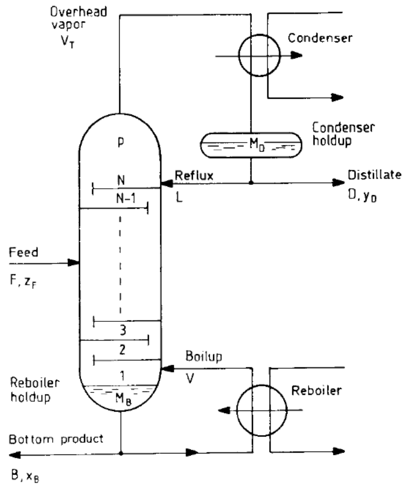


Figure 13–6. Automatic control system for Perco motor fuel alkylation process.

Example system: Distillation column



Raw oil inserted at bottom; different petro-chemical subcomponents extracted

Example system: Distillation column

Outputs:

y_1 = top draw composition

y_2 = side draw composition

Inputs:

u_1 = top draw flowrate

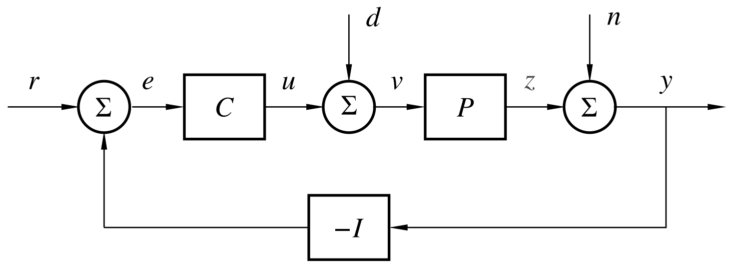
u_2 = side draw flowrate

u_3 = bottom temperature control input

Linear first-order plus deadtime (FOPDT) model:

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{4}{50s+1}e^{-27s} & \frac{1.8}{60s+1}e^{-28s} & \frac{5.9}{50s+1}e^{-27s} \\ \frac{5.4}{50s+1}e^{-18s} & \frac{5.7}{60s+1}e^{-14s} & \frac{6.9}{40s+1}e^{-15s} \end{bmatrix}}_{P(s)} \begin{bmatrix} U_1(s) \\ U_2(s) \\ U_3(s) \end{bmatrix}$$

Multivariable transfer functions



P and C are matrices and all signals are vectors – order matters!

$$Z = PCR + PD - PC(N + Z)$$

$$(I + PC)Z = PCR + PD - PCN$$

$$Z = \underbrace{(I + PC)^{-1}PC}_{G_{zr}=T}\ R + \underbrace{(I + PC)^{-1}P}_{G_{zd}}\ D - \underbrace{(I + PC)^{-1}PC}_{G_{zn}}\ N$$

Sensitivity functions for MIMO systems

Output sensitivity function:

$$(I + PC)^{-1} = S$$

Input sensitivity function:

$$(I + CP)^{-1}$$

Mini-problem:

Find the sensitivity functions above in the block diagram on the previous slide.

Some useful identities

Notice the following identities:

$$(i)\quad [I + PC]^{-1}P = P[I + CP]^{-1}$$

$$(ii)\quad C[I + PC]^{-1} = [I + CP]^{-1}C$$

$$(iii)\quad T = P[I + CP]^{-1}C = PC[I + PC]^{-1} = [I + PC]^{-1}PC$$

$$(iv)\quad S + T = I$$

Proof:

The first equality follows by multiplication on both sides with $[I + PC]$ from the left and with $[I + CP]$ from the right.

Left: $[I + PC][I + PC]^{-1}P[I + CP] = I \cdot [P + PCP] = [I + PC]P$

Right: $[I + PC]P[I + CP]^{-1}[I + CP] = [I + PC]P \cdot I = [I + PC]P$

Lecture 7 – Outline

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Hard limitations from RHP zeros

THEOREM:

Assume that the MIMO system $P(s)$ has a transmission zero z in the RHP.

Let $S(s) = [I + P(s)C(s)]^{-1}$ and let $W_S(s)$ be a scalar, stable and minimum phase transfer function. Then the specification

$$\|W_S S\|_\infty = \sup_\omega \overline{\sigma}\big(W_S(i\omega)S(i\omega)\big) \le 1$$

is possible to meet **only if**

$$|W_S(z)| \le 1$$

Example: Control of MIMO system with RHP zero

Recall the following process from Lecture 6:

$$P(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}$$

Computing the determinant

$$\det P(s) = \frac{2}{(s+1)^2} - \frac{3}{(s+2)(s+1)} = \frac{-s+1}{(s+1)^2(s+2)}$$

shows that the process has a RHP zero in 1, which will limit the achievable performance.

[See lecture notes for details of the following slides]

Example – Controller 1

The controller

$$C_1(s) = \begin{bmatrix} \frac{K_1(s+1)}{s} & -\frac{3K_2(s+0.5)}{s(s+2)} \\ -\frac{K_1(s+1)}{s} & \frac{2K_2(s+0.5)}{s(s+1)} \end{bmatrix}$$

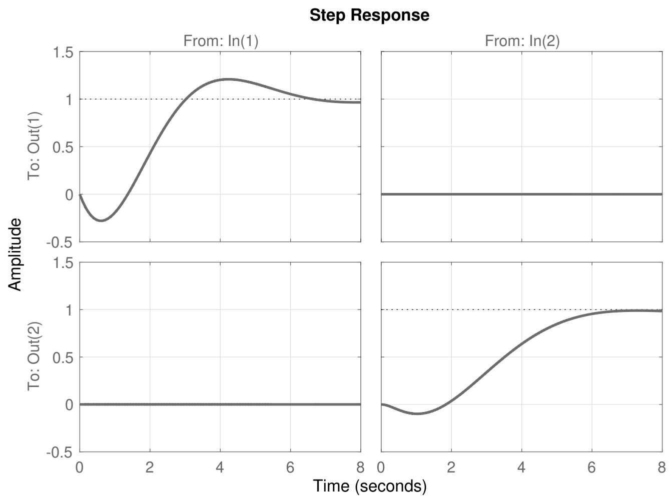
gives the diagonal loop transfer matrix

$$P(s)C_1(s) = \begin{bmatrix} \frac{K_1(-s+1)}{s(s+2)} & 0 \\ 0 & \frac{K_2(s+0.5)(-s+1)}{s(s+1)(s+2)} \end{bmatrix}$$

The system is decoupled into two scalar loops, each with an unstable zero at $s = 1$ that limits the bandwidth.

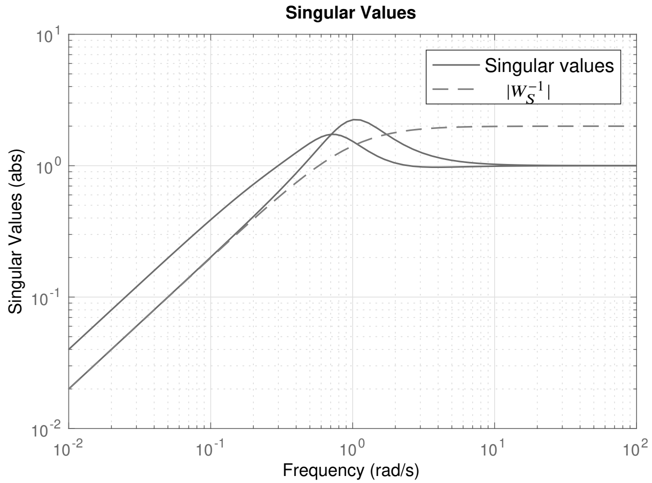
Closed-loop step responses from (r_1, r_2) to (y_1, y_2) for $K_1 = K_2 = 1$ are shown on next slide.

Step responses using Controller 1



No cross-coupling, but RHP zero shows up in both y_1 and y_2.

Sensitivity sigma plot using Controller 1



$W_S(s) = \frac{s+1.01}{2s}$, impossible to meet due to RHP zero

Example – Controller 2

The controller

$$C_2(s) = \begin{bmatrix} \frac{K_1(s+1)}{s} & K_2 \\ -\frac{K_1(s+1)}{s} & K_2 \end{bmatrix}$$

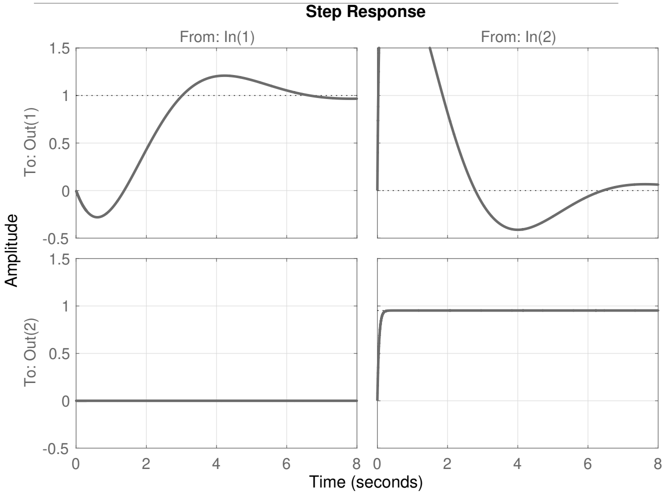
gives the triangular loop transfer matrix

$$P(s)C_2(s) = \begin{bmatrix} \frac{K_1(-s+1)}{s(s+2)} & \frac{K_2(5s+7)}{(s+2)(s+1)} \\ 0 & \frac{2K_2}{s+1} \end{bmatrix}$$

Now the decoupling is only partial: Output y_2 is not affected by r_1. Moreover, no RHP zero limits the rate of response in y_2!

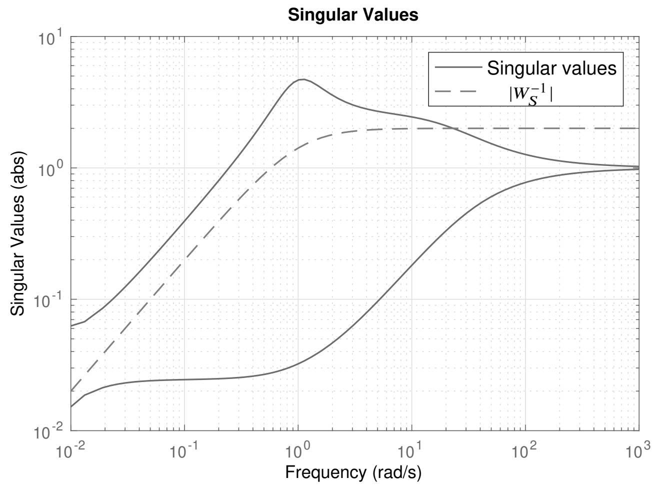
The closed-loop step responses for $K_1 = 1$, $K_2 = 10$ are shown on next slide.

Step responses using Controller 2



The RHP zero does not prevent a fast y_2 response to r_2 but at the price of a simultaneous undesired response in y_1.

Sensitivity sigma plot using Controller 2



$W_S(s) = \frac{s+1.01}{2s}$, impossible to meet due to RHP zero

Example – Controller 3

The controller

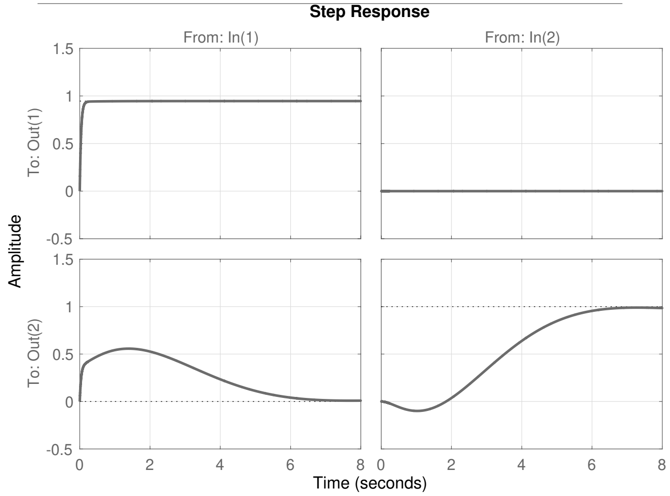
$$C_3(s) = \begin{bmatrix} K_1 & \frac{-3K_2(s+0.5)}{s(s+2)} \\ K_1 & \frac{2K_2(s+0.5)}{s(s+1)} \end{bmatrix}$$

gives the triangular loop transfer matrix

$$P(s)C_3(s) = \begin{bmatrix} \frac{K_1(5s+7)}{(s+1)(s+2)} & 0 \\ \frac{2K_1}{s+1} & \frac{K_2(-1+s)(s+0.5)}{s(s+1)^2(s+2)} \end{bmatrix}$$

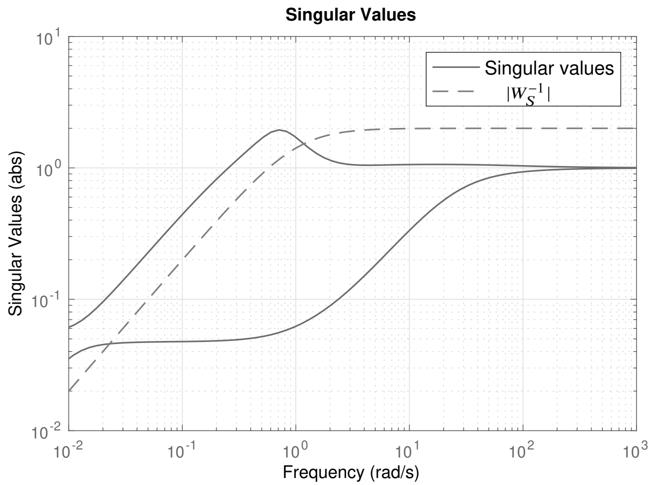
In this case y_1 is decoupled from r_2 and can respond arbitrarily fast for high values of K_1, at the expense of bad behavior in y_2. Step responses for $K_1 = 10$, $K_2 = 1$ are shown on next slide.

Step responses using Controller 3



The RHP zero does not prevent a fast y_1 response to r_1 but at the price of a simultaneous undesired response in y_2.

Sensitivity sigma plot using Controller 3



$W_S(s) = \frac{s+1.01}{2s}$, impossible to meet due to RHP zero

Example – summary

To summarize, the example shows that even though a **multivariable RHP zero always gives a performance limitation**, it is possible to influence where the effects should show up.

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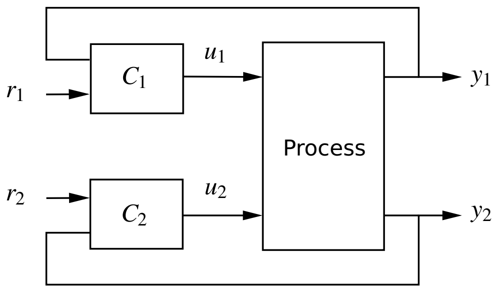
Decentralized control

Background in process control:

- A few important variables were controlled using the simple loop paradigm: one sensor, one actuator, one controller
- As more loops were added, interaction was handled using feedforward, cascade and midrange control, selectors, etc.
- Not always obvious how to associate sensors and actuators – the pairing problem

Computer control and state-space design methods eventually led to centralized MIMO control schemes (LQG, MPC, etc.)

Interaction between simple loops



$$Y_1(s) = P_{11}(s)U_1(s) + P_{12}U_2(s)$$
$$Y_2(s) = P_{21}(s)U_1(s) + P_{22}U_2(s),$$

What happens when the controllers are tuned individually (C_1 for P_{11} and C_2 for P_{22}), ignoring the cross-couplings (P_{12} and P_{21})?

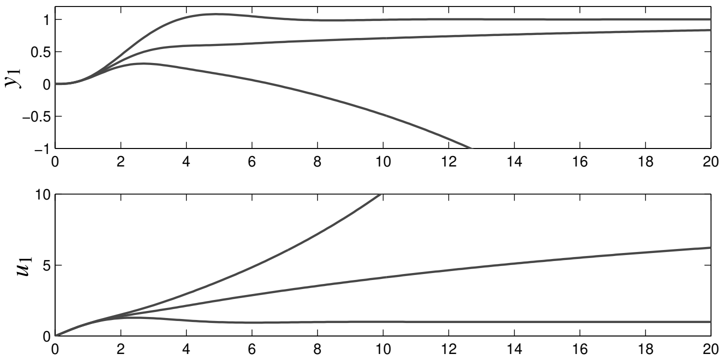
Rosenbrock's example

$$P(s) = \begin{pmatrix} \frac{1}{(s+1)^2} & \frac{2}{(s+1)^2} \\ \frac{1}{(s+1)^2} & \frac{1}{(s+1)^2} \end{pmatrix}$$

Very benign subsystems, no fundamental limitations.

Rosenbrock's example with two SISO controllers

- $U_1 = \left(1 + \frac{1}{s}\right)(R_1 - Y_1)$
- $U_2 = -K_2 Y_2$ with $K_2 = 0$, 0.8, and 1.6.



The second controller has a major impact on the first loop! Gain reversal in $u_1 \rightarrow y_1$ when $K_2 = 1.6$.

Bristol's Relative Gain Array (RGA)

- Edgar H. Bristol, "On a new measure of interaction for multivariable process control" [*IEEE TAC* 11(1967) pp. 133–135]
- A simple way of measuring interaction in MIMO systems
- Idea: Study how the gain between one input and one output changes when all other outputs are regulated:

$$\text{relative gain} = \frac{\text{open-loop gain}}{\text{“closed-loop gain”}}$$

- Often only the static gain $P(0)$ is analyzed, but one could also look at for instance $P(i\omega_c)$ and other frequencies

Calculation of RGA

Assume the input-output relation $y = Gu$, where G is square and invertible.

Open loop: Assume $u_j \neq 0$ and all other inputs zero. Then

$$y_k = G_{kj} u_j$$

Closed loop: Assume $y_k \neq 0$ and that all other outputs are regulated to zero. Solving for the corresponding inputs gives

$$u_j = G^{-1}_{jk} y_k \quad \Leftrightarrow \quad y_k = \frac{1}{G^{-1}_{jk}} u_j$$

Calculation of RGA

Relative gain:

$$\lambda_{kj} = G_{kj} \cdot G^{-1}_{jk}$$

All elements of the relative gain array (matrix) can be computed in one go as

$$\Lambda = \text{RGA}(G) = G \mathbin{.*} (G^{-1})^T$$

where $.*$ denotes element-wise (Hadamard/Schur) multiplication

Matlab: `RGA = G.*inv(G).'`

Properties of RGA

- RGA is dimensionless; not affected by choice of units or scaling.
- RGA is normalized: Rows and columns of Λ sum to 1.
- Diagonal or triangular plant gives $\Lambda = I$.

Interpretation of RGA

- $\lambda_{kj} \approx 1$ means small closed-loop interaction. Suitable to pair output k with input j.
- $\lambda_{kj} < 0$ corresponds to a sign reversal due to feedback and a risk of instability if output k is paired with input j – avoid!
- $0 < \lambda_{kj} < 1$ means that the closed-loop gain is larger than the open-loop gain; the opposite is true for $\lambda_{kj} > 1$.

Rule of thumb: Pair the outputs and inputs so that corresponding relative gains are positive and as close to 1 as possible.

RGA of Rosenbrock's example

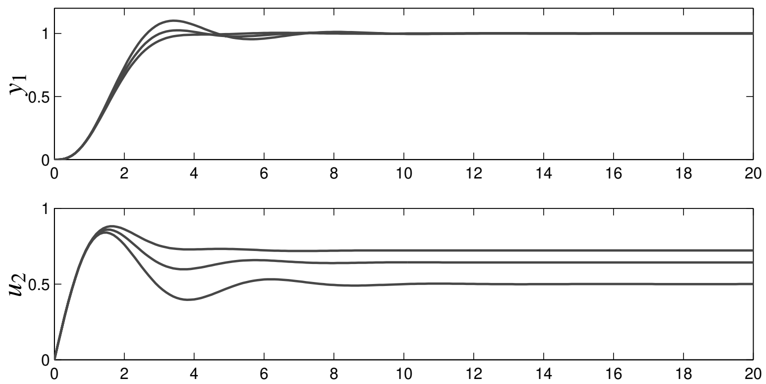
Analysis of static gain:

$$P(0) = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \qquad P^{-1}(0) = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

$$\Lambda = P(0) \,.\!*\, (P^{-1}(0))^T = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$$

- Negative value of λ_{11} indicates the problematic sign reversal found previously when y_1 was controlled using u_1.
- Better to use reverse pairing, i.e. let u_2 control y_1 and vice versa.

Rosenbrock's example with reverse pairing



- $U_2 = \left(1 + \frac{1}{s}\right)(R_1 - Y_1)$
- $U_1 = -K_2 Y_2$ with $K_2 = 0, 0.8$, and 1.6.

RGA of non-square systems

The RGA can also be computed for a general gain matrix G:

$$\mathrm{RGA}(G) = G \mathbin{.*} \left(G^{\dagger}\right)^T$$

Here, † denotes the pseudo-inverse (Matlab: `pinv`)

Example: Distillation column:

$$P(0) = \begin{pmatrix} 4.0 & 1.8 & 5.9 \\ 5.4 & 5.7 & 6.9 \end{pmatrix}, \quad \mathrm{RGA}(P(0)) = \begin{pmatrix} 0.28 & -0.61 & 1.33 \\ 0.01 & 1.58 & -0.59 \end{pmatrix}$$

Suggested pairing for decentralized control: y_1—u_3, y_2—u_2, u_1 unused

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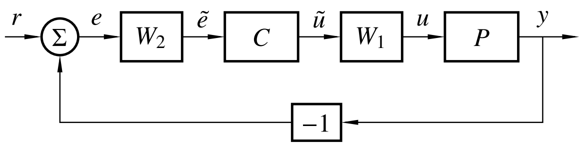
Decoupling



Idea: Select decoupling filters W_1 and W_2 so that the controller sees a diagonal plant:

$$\tilde{P} = W_2 P W_1 = \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$$

Then we can use a decentralized controller C with the same diagonal structure.

Decoupling

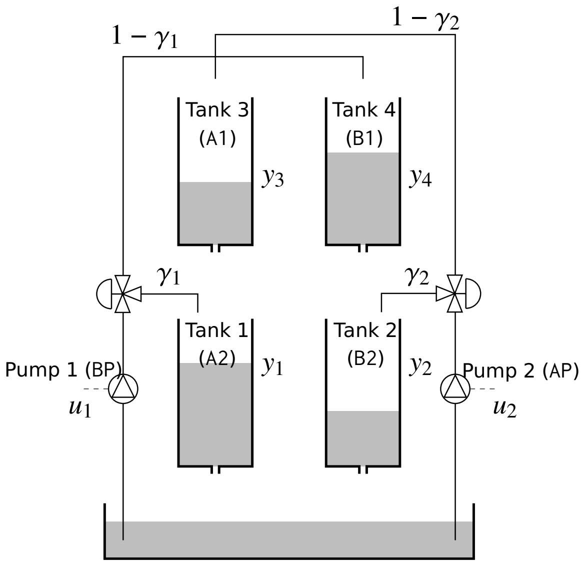
Many variants/names:

- Input/conventional/feedforward decoupling: $\tilde{P} = PW_1,\ W_2 = I$
- Output/inverse/feedback decoupling: $\tilde{P} = W_2 P,\ W_1 = I$

W_1 and W_2 can be static or dynamic systems

Example: Static input decoupling: $W_1 = P^{-1}(0),\ W_2 = I$

Lab 2: The quadruple tank



Summary

- All real MIMO systems are coupled
- Multivariable RHP zeros $\Rightarrow$ limitations
  - Don't forget process redesign
- Decentralized control – one controller per controlled variable
  - RGA gives insight for input–output pairing
- Decoupling
  - Simpler system
  - SISO design, tuning and operation can be used