LUNDS UNIVERSITET

# Lecture 8

## FRTN10 Multivariable Control

Automatic Control LTH, 2018

# Course Outline

L1–L5 Specifications, models and loop-shaping by hand

L6–L8 Limitations on achievable performance

6. Controllability/observability, multivariable poles/zeros
7. Fundamental limitations
8. **Multivariable and decentralized control**

L9–L11 Controller optimization: analytic approach

L12–L14 Controller optimization: numerical approach

L15 Course review

# Lecture 8 – Outline

1. Transfer functions for MIMO systems
2. Limitations due to RHP zeros
3. Decentralized control
4. Decoupling

# Typical process control system

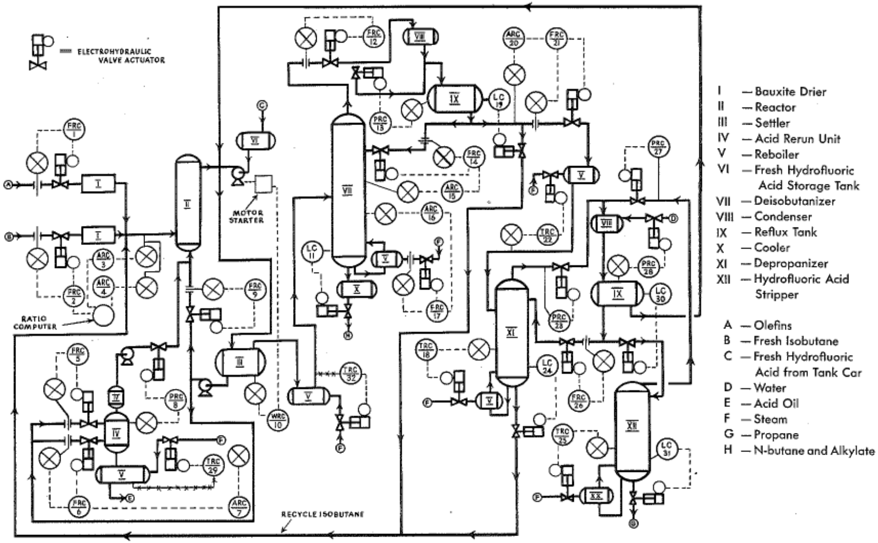
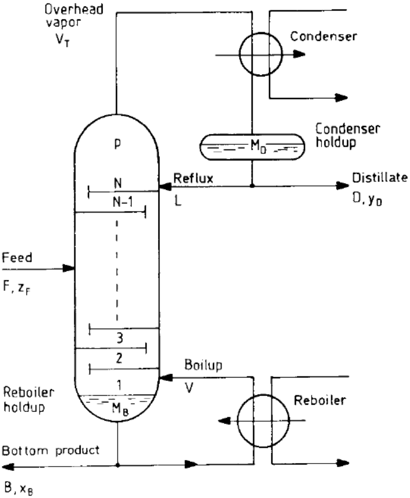



Figure 13–6. Automatic control system for Perco motor fuel alkylation process.

# Example system: Distillation column




Raw oil inserted at bottom; different petro-chemical subcomponents extracted

# Example system: Distillation column

**Outputs:**

$y_1$ = top draw composition

$y_2$ = side draw composition

**Inputs:**

$u_1$ = top draw flowrate

$u_2$ = side draw flowrate

$u_3$ = bottom temperature control input

Linear first-order plus deadtime (FOPDT) model:

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{4}{50s+1}e^{-27s} & \frac{1.8}{60s+1}e^{-28s} & \frac{5.9}{50s+1}e^{-27s} \\ \frac{5.4}{50s+1}e^{-18s} & \frac{5.7}{60s+1}e^{-14s} & \frac{6.9}{40s+1}e^{-15s} \end{bmatrix}}_{P(s)} \begin{bmatrix} U_1(s) \\ U_2(s) \\ U_3(s) \end{bmatrix}$$

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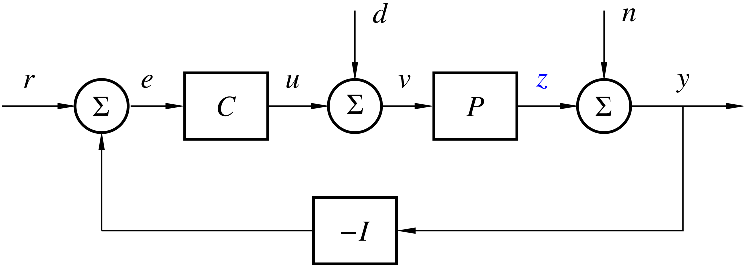
# Multivariable transfer functions



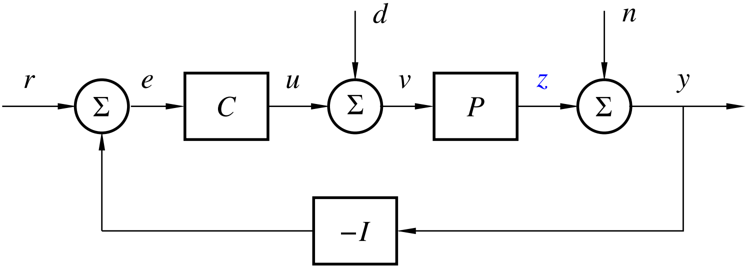

$P$ and $C$ are matrices and all signals are vectors – order matters!

$$Z = PCR + PD - PC(N + Z)$$

$$(I + PC)Z = PCR + PD - PCN$$

$$Z = \underbrace{(I + PC)^{-1}PC}_{G_{zr}=T}\,R + \underbrace{(I + PC)^{-1}P}_{G_{zd}}\,D - \underbrace{(I + PC)^{-1}PC}_{G_{zn}}\,N$$

# Multivariable transfer functions

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$$Z = \underbrace{(I + PC)^{-1}PC}_{G_{zr}=T}\; R + \underbrace{(I + PC)^{-1}P}_{G_{zd}}\; D - \underbrace{(I + PC)^{-1}PC}_{G_{zn}}\; N$$

# Sensitivity functions for MIMO systems

Output sensitivity function:

$$(I + PC)^{-1} = S$$

Input sensitivity function:

$$(I + CP)^{-1}$$

**Mini-problem**:

Find the sensitivity functions above in the block diagram on the previous slide.

# Some useful identities

Notice the following identities:

$$(i)\ [I + PC]^{-1}P = P[I + CP]^{-1}$$

$$(ii)\ C[I + PC]^{-1} = [I + CP]^{-1}C$$

$$(iii)\ T = P[I + CP]^{-1}C = PC[I + PC]^{-1} = [I + PC]^{-1}PC$$

$$(iv)\ S + T = I$$

Proof:

The first equality follows by multiplication on both sides with $[I + PC]$ from the left and with $[I + CP]$ from the right.

Left: $[I + PC][I + PC]^{-1}P[I + CP] = I \cdot [P + PCP] = [I + PC]P$

Right: $[I + PC]P[I + CP]^{-1}[I + CP] = [I + PC]P \cdot I = [I + PC]P$

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$$(iii)\quad T = P[I+CP]^{-1}C = PC[I+PC]^{-1} = [I+PC]^{-1}PC$$

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# Hard limitations from RHP zeros

THEOREM:

Assume that the MIMO system $P(s)$ has a transmission zero $z$ in the RHP.

Let $S(s) = [I + P(s)C(s)]^{-1}$ and let $W_S(s)$ be a scalar, stable and minimum phase transfer function. Then the specification

$$\|W_S S\|_\infty = \sup_\omega \overline{\sigma}\big(W_S(i\omega)S(i\omega)\big) \leq 1$$

is possible to meet **only if**

$$|W_S(z)| \leq 1$$

# Example: Control of MIMO system with RHP zero

Recall the following process from Lecture 6:

$$P(s) = \begin{bmatrix} \dfrac{2}{s+1} & \dfrac{3}{s+2} \\ \dfrac{1}{s+1} & \dfrac{1}{s+1} \end{bmatrix}$$

Computing the determinant

$$\det P(s) = \frac{2}{(s+1)^2} - \frac{3}{(s+2)(s+1)} = \frac{-s+1}{(s+1)^2(s+2)}$$

shows that the process has a RHP zero in $1$, which will limit the achievable performance.

[See lecture notes for details of the following slides]

# Example – Controller 1

The controller

$$C_1(s) = \begin{bmatrix} \frac{K_1(s+1)}{s} & -\frac{3K_2(s+0.5)}{s(s+2)} \\ -\frac{K_1(s+1)}{s} & \frac{2K_2(s+0.5)}{s(s+1)} \end{bmatrix}$$

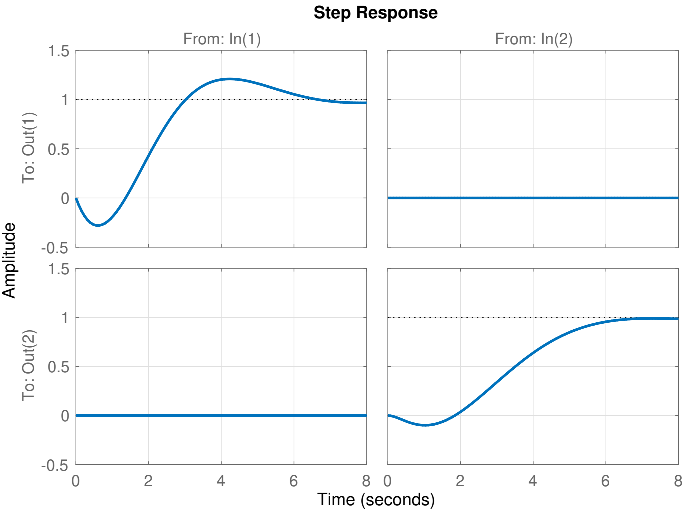
gives the diagonal loop transfer matrix

$$P(s)C_1(s) = \begin{bmatrix} \frac{K_1(-s+1)}{s(s+2)} & 0 \\ 0 & \frac{K_2(s+0.5)(-s+1)}{s(s+1)(s+2)} \end{bmatrix}$$

The system is decoupled into two scalar loops, each with an unstable zero at $s = 1$ that limits the bandwidth.

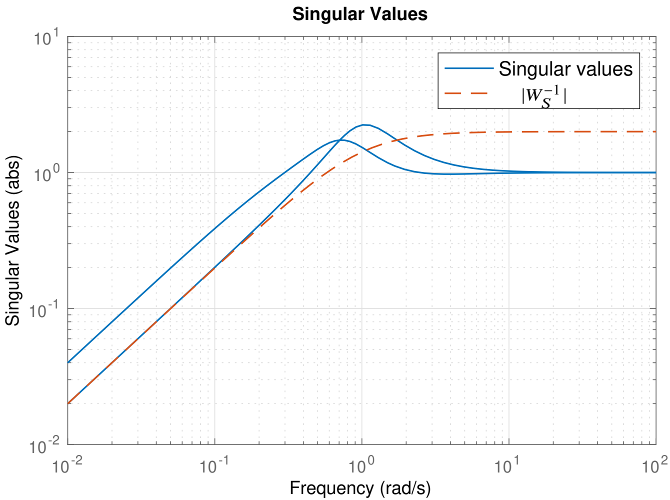
Closed-loop step responses from $(r_1, r_2)$ to $(y_1, y_2)$ for $K_1 = K_2 = 1$ are shown on next slide.

# Step responses using Controller 1




No cross-coupling, but RHP zero shows up in both $y_1$ and $y_2$.

# Sensitivity sigma plot using Controller 1




$W_S(s) = \frac{s+1.01}{2s}$, impossible to meet due to RHP zero

# Example – Controller 2

The controller

$$C_2(s) = \begin{bmatrix} \frac{K_1(s+1)}{s} & K_2 \\ -\frac{K_1(s+1)}{s} & K_2 \end{bmatrix}$$

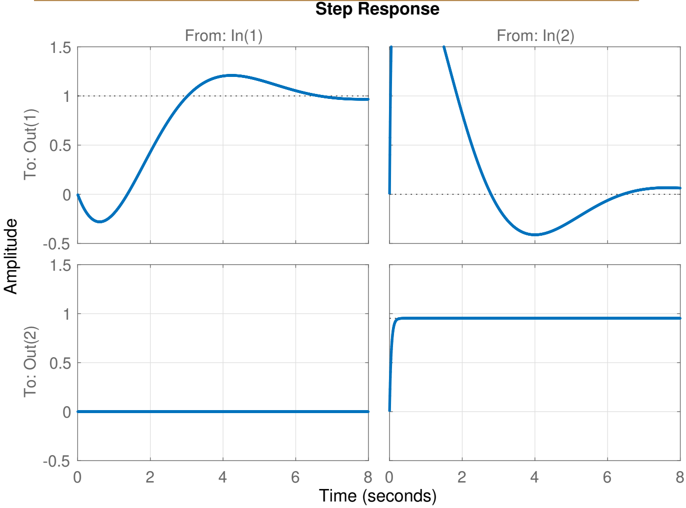
gives the triangular loop transfer matrix

$$P(s)C_2(s) = \begin{bmatrix} \frac{K_1(-s+1)}{s(s+2)} & \frac{K_2(5s+7)}{(s+2)(s+1)} \\ 0 & \frac{2K_2}{s+1} \end{bmatrix}$$

Now the decoupling is only partial: Output $y_2$ is not affected by $r_1$. Moreover, no RHP zero limits the rate of response in $y_2$!

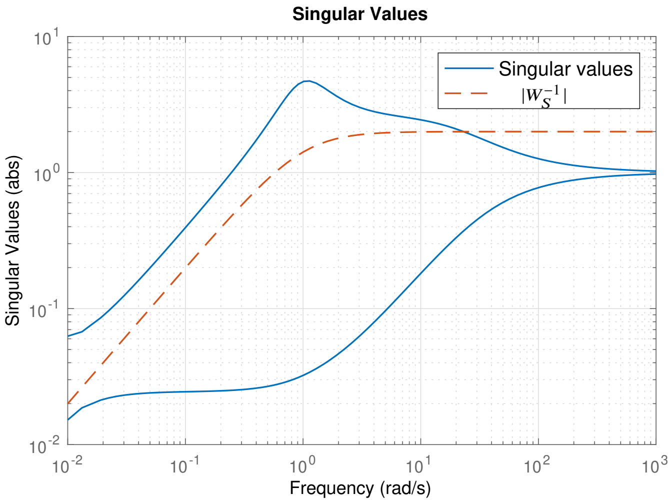
The closed-loop step responses for $K_1 = 1$, $K_2 = 10$ are shown on next slide.

# Step responses using Controller 2




The RHP zero does not prevent a fast $y_2$ response to $r_2$ but at the price of a simultaneous undesired response in $y_1$.

# Sensitivity sigma plot using Controller 2




$W_S(s) = \frac{s+1.01}{2s}$, impossible to meet due to RHP zero

# Example – Controller 3

The controller

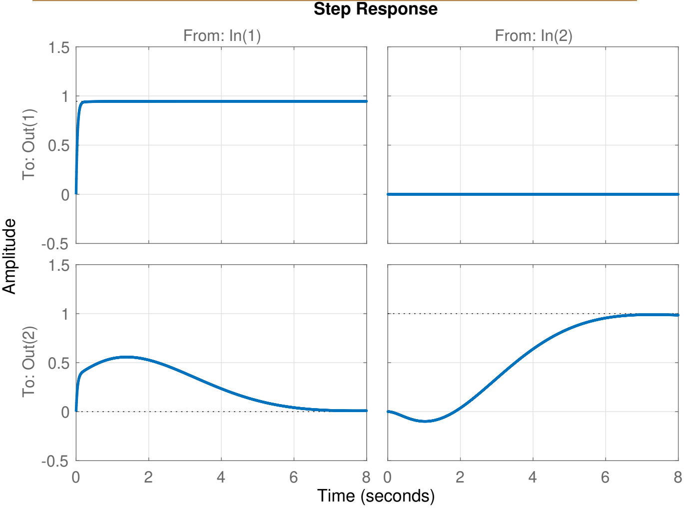
$$C_3(s) = \begin{bmatrix} K_1 & \frac{-3K_2(s+0.5)}{s(s+2)} \\ K_1 & \frac{2K_2(s+0.5)}{s(s+1)} \end{bmatrix}$$

gives the triangular loop transfer matrix

$$P(s)C_3(s) = \begin{bmatrix} \frac{K_1(5s+7)}{(s+1)(s+2)} & 0 \\ \frac{2K_1}{s+1} & \frac{K_2(-1+s)(s+0.5)}{s(s+1)^2(s+2)} \end{bmatrix}$$

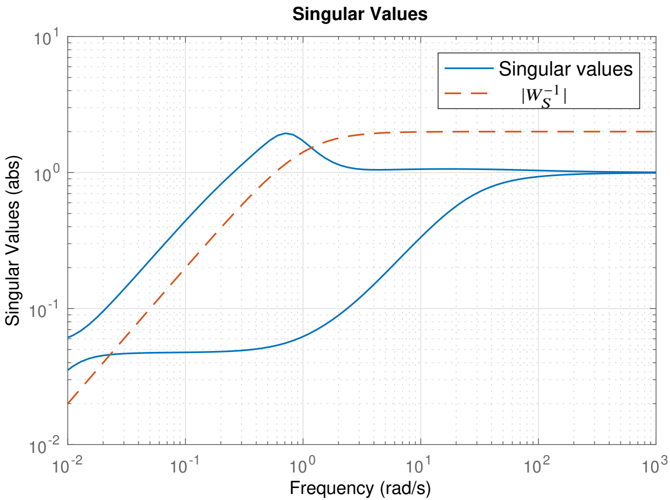
In this case $y_1$ is decoupled from $r_2$ and can respond arbitrarily fast for high values of $K_1$, at the expense of bad behavior in $y_2$. Step responses for $K_1 = 10$, $K_2 = 1$ are shown on next slide.

# Step responses using Controller 3




The RHP zero does not prevent a fast $y_1$ response to $r_1$ but at the price of a simultaneous undesired response in $y_2$.

# Sensitivity sigma plot using Controller 3




$W_S(s) = \frac{s+1.01}{2s}$, impossible to meet due to RHP zero

# Example – summary

To summarize, the example shows that even though a **multivariable RHP zero always gives a performance limitation**, it is possible to influence where the effects should show up.

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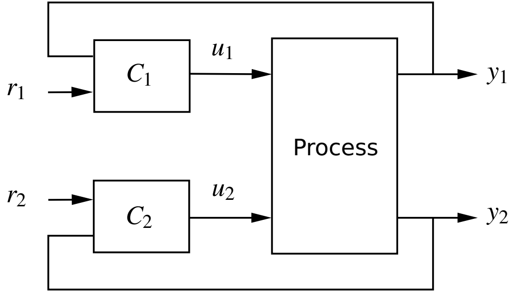
# Decentralized control

Background in process control:

- A few important variables were controlled using the simple loop paradigm: one sensor, one actuator, one controller
- As more loops were added, interaction was handled using feedforward, cascade and midrange control, selectors, etc.
- Not always obvious how to associate sensors and actuators – the pairing problem

Computer control and state-space design methods eventually led to centralized MIMO control schemes (LQG, MPC, etc.)

# Interaction between simple loops




$$Y_1(s) = P_{11}(s)U_1(s) + P_{12}U_2(s)$$
$$Y_2(s) = P_{21}(s)U_1(s) + P_{22}U_2(s),$$

What happens when the controllers are tuned individually ($C_1$ for $P_{11}$ and $C_2$ for $P_{22}$), ignoring the cross-couplings ($P_{12}$ and $P_{21}$)?

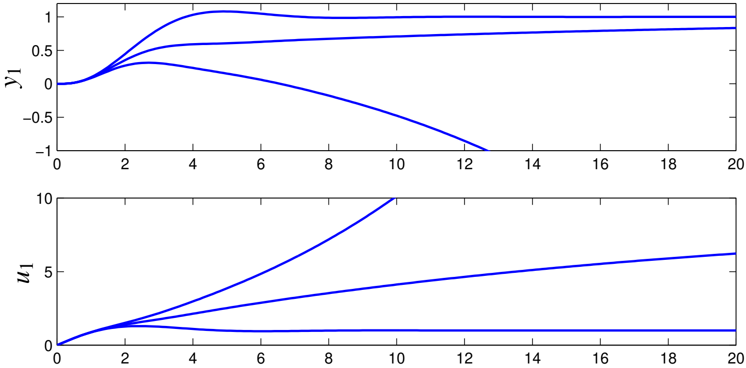
# Rosenbrock's example

$$P(s) = \begin{pmatrix} \frac{1}{(s+1)^2} & \frac{2}{(s+1)^2} \\ \frac{1}{(s+1)^2} & \frac{1}{(s+1)^2} \end{pmatrix}$$

Very benign subsystems, no fundamental limitations.

# Rosenbrock's example with two SISO controllers

- $U_1 = \left(1 + \frac{1}{s}\right)(R_1 - Y_1)$
- $U_2 = -K_2 Y_2$ with $K_2 = 0, 0.8$, and $1.6$.




The second controller has a major impact on the first loop! Gain reversal in $u_1 \to y_1$ when $K_2 = 1.6$.

# Bristol's Relative Gain Array (RGA)

- Edgar H. Bristol, "On a new measure of interaction for multivariable process control" [*IEEE TAC* 11(1967) pp. 133–135]
- A simple way of measuring interaction in MIMO systems
- Idea: Study how the gain between one input and one output changes when all other outputs are regulated:

$$\text{relative gain} = \frac{\text{open-loop gain}}{\text{“closed-loop gain”}}$$

- Often only the static gain $P(0)$ is analyzed, but one could also look at for instance $P(i\omega_c)$ and other frequencies

# Calculation of RGA

Assume the input-output relation $y = Gu$, where $G$ is square and invertible.

**Open loop:** Assume $u_j \neq 0$ and all other inputs zero. Then

$$y_k = G_{kj} u_j$$

**Closed loop:** Assume $y_k \neq 0$ and that all other outputs are regulated to zero. Solving for the corresponding inputs gives

$$u_j = G^{-1}_{jk} y_k \quad \Leftrightarrow \quad y_k = \frac{1}{G^{-1}_{jk}} u_j$$

# Calculation of RGA

Relative gain:

$$\lambda_{kj} = G_{kj} \cdot G^{-1}_{jk}$$

All elements of the relative gain array (matrix) can be computed in one go as

$$\Lambda = \mathrm{RGA}(G) = G \mathbin{.*} (G^{-1})^T$$

where $.*$ denotes element-wise (Hadamard/Schur) multiplication

Matlab: `RGA = G.*inv(G).'`

# Properties of RGA

- RGA is dimensionless; not affected by choice of units or scaling.
- RGA is normalized: Rows and columns of $\Lambda$ sum to 1.
- Diagonal or triangular plant gives $\Lambda = I$.

# Interpretation of RGA

- $\lambda_{kj} \approx 1$ means small closed-loop interaction. Suitable to pair output $k$ with input $j$.
- $\lambda_{kj} < 0$ corresponds to a sign reversal due to feedback and a risk of instability if output $k$ is paired with input $j$ – avoid!
- $0 < \lambda_{kj} < 1$ means that the closed-loop gain is larger than the open-loop gain; the opposite is true for $\lambda_{kj} > 1$.

**Rule of thumb:** Pair the outputs and inputs so that corresponding relative gains are positive and as close to $1$ as possible.

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# RGA of Rosenbrock's example

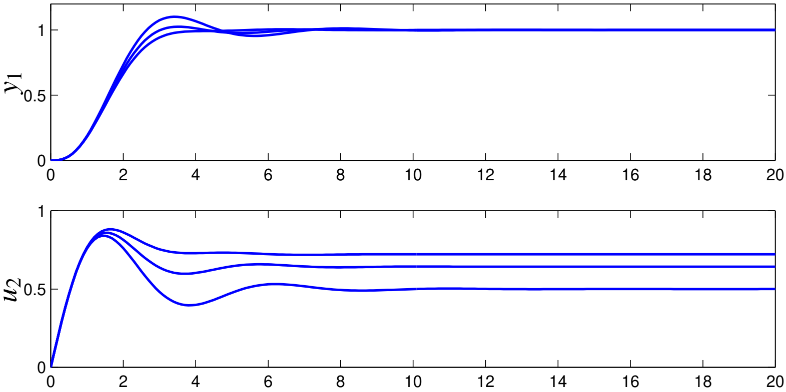
Analysis of static gain:

$$P(0) = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \qquad P^{-1}(0) = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

$$\Lambda = P(0) .* (P^{-1}(0))^T = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$$

- Negative value of $\lambda_{11}$ indicates the problematic sign reversal found previously when $y_1$ was controlled using $u_1$.
- Better to use reverse pairing, i.e. let $u_2$ control $y_1$ and vice versa.

# Rosenbrock's example with reverse pairing




- $U_2 = \left(1 + \frac{1}{s}\right)(R_1 - Y_1)$
- $U_1 = -K_2 Y_2$ with $K_2 = 0, 0.8$, and $1.6$.

# RGA of non-square systems

The RGA can also be computed for a general gain matrix $G$:

$$\mathrm{RGA}(G) = G \,.\!*\, \left(G^{\dagger}\right)^{T}$$

Here, † denotes the pseudo-inverse (Matlab: `pinv`)

**Example:** Distillation column:

$$P(0) = \begin{pmatrix} 4.0 & 1.8 & 5.9 \\ 5.4 & 5.7 & 6.9 \end{pmatrix}, \quad \mathrm{RGA}(P(0)) = \begin{pmatrix} 0.28 & -0.61 & 1.33 \\ 0.01 & 1.58 & -0.59 \end{pmatrix}$$

Suggested pairing for decentralized control: $y_1$—$u_3$, $y_2$—$u_2$, $u_1$ unused

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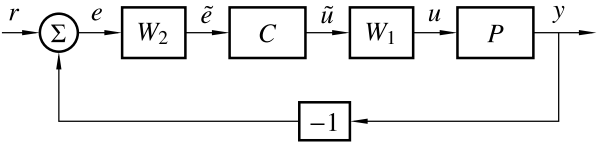
# Decoupling




Idea: Select decoupling filters $W_1$ and $W_2$ so that the controller sees a diagonal plant:

$$\tilde{P} = W_2 P W_1 = \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$$

Then we can use a decentralized controller $C$ with the same diagonal structure.

# Decoupling

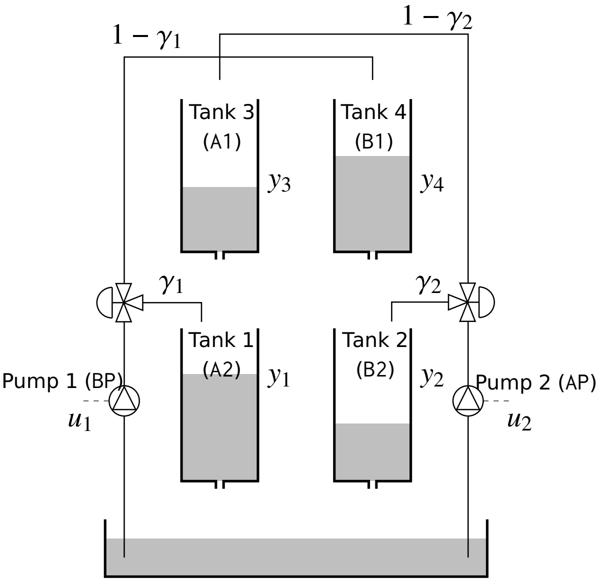
Many variants/names:

- Input/conventional/feedforward decoupling: $\tilde{P} = PW_1, \; W_2 = I$
- Output/inverse/feedback decoupling: $\tilde{P} = W_2 P, \; W_1 = I$

$W_1$ and $W_2$ can be static or dynamic systems

**Example:** Static input decoupling: $W_1 = P^{-1}(0), \; W_2 = I$

# Lab 2: The quadruple tank



# Summary

- All real MIMO systems are coupled
- Multivariable RHP zeros $\Rightarrow$ limitations
  - Don't forget process redesign
- Decentralized control – one controller per controlled variable
  - RGA gives insight for input–output pairing
- Decoupling

  Simpler system

  SISO design, tuning and operation can be used