Lecture 7 – Outline

1. Bode's Relation and Bode's Integral
2. Limitations from RHP poles/zeros and delays: insights from loop shaping
3. Limitations from RHP poles/zeros: Hard proofs

Limitations in control design

What we already know:

- Model uncertainty, measurement noise, and control signal limitations give upper limits on the achievable bandwidth
- $S + T = 1$, which implies

$$|S(i\omega)| + |T(i\omega)| \geq 1$$
$$\big||S(i\omega)| - |T(i\omega)|\big| \leq 1$$

- Some modes may be impossible to control or observe due to lack of controllability or observability

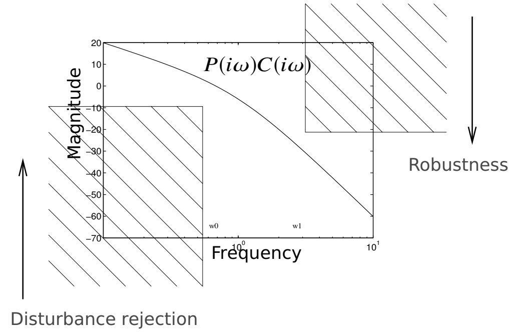
Limitations in control design

Fundamental limitations:

- Bode's Relation: amplitude and phase are coupled
- Bode's Integral: $|S(i\omega)|$ (and $|T(i\omega)|$) cannot be made small everywhere
- Limitations from non-minimum-phase elements:
  - unstable poles
  - right-half-plane (RHP) zeros
  - time delays

Recall: Loop shaping

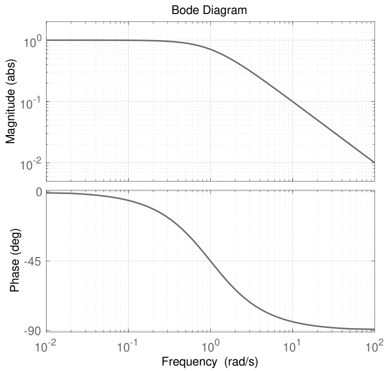
The loop transfer function $L = PC$ should be made large at low frequencies and small at high frequencies:



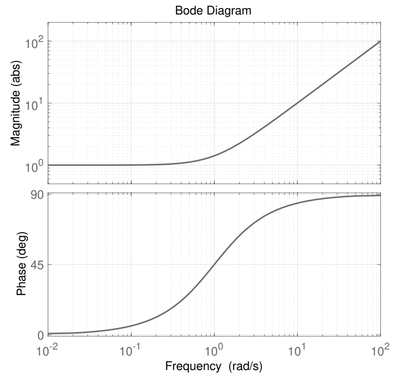
How quickly can we make the transition from high to low gain and still retain a good phase margin?

Recall: Amplitude and phase are coupled

LHP (stable) pole



LHP (“stable”) zero



If $G(s)$ is minimum phase (no RHP poles/zeros or time delays) then

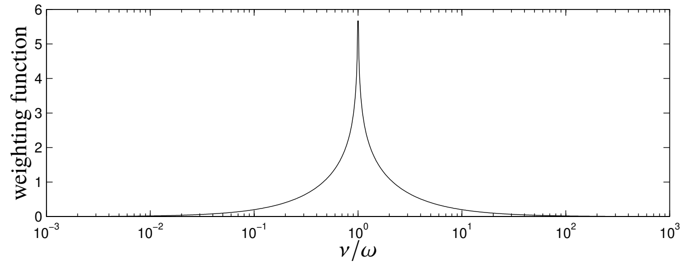
$$\arg G(i\omega) \approx \frac{\pi}{2} \frac{d \log |G(i\omega)|}{d \log \omega}$$

Bode’s Relation

If $G(s)$ is minimum phase, then

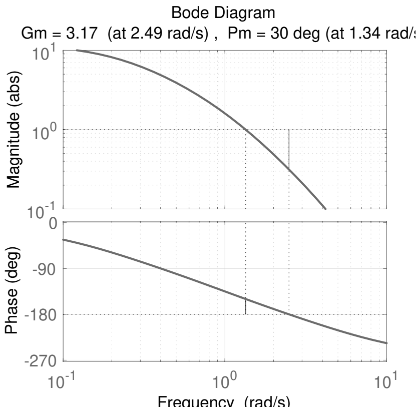
$$\arg G(i\omega) = \frac{2\omega}{\pi} \int_0^\infty \frac{\log |G(i\nu)| - \log |G(i\omega)|}{\nu^2 - \omega^2} d\nu$$

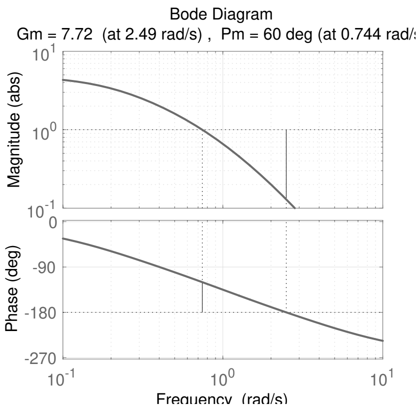
$$= \frac{1}{\pi} \int_0^\infty \frac{d \log |G(i\nu)|}{d \log \nu} \underbrace{\log \left| \frac{\nu + \omega}{\nu - \omega} \right|}_{\text{weighting function}} d \log \nu$$



Consequence for phase margin

For minimum-phase systems, to have a phase margin between 30° and 60°, the slope of the amplitude curve should be between approx. -1.67 and -1.33 at the cross-over frequency.





Bode’s Integral – stable system

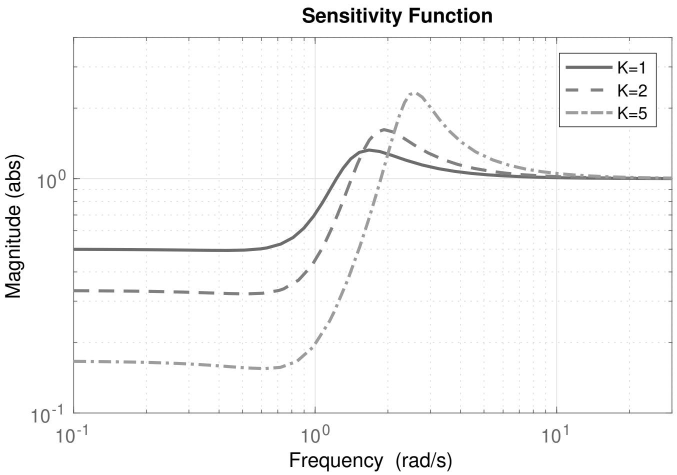
For a stable system with loop gain $L(s)$ with relative degree ≥ 2 the following *conservation law* for the sensitivity function $S(s) = (1 + L(s))^{-1}$ holds:

$$\int_0^\infty \log |S(i\omega)| d\omega = 0$$

(Sometimes known as the "waterbed effect")

Example

P-control of $(s^2 + s + 1)^{-1}$



Bode's Integral – general case

For a system with loop gain with relative degree ≥ 2 and unstable poles $p_1, \ldots, p_M$, the following *conservation law* for the sensitivity function holds:

$$\int_0^\infty \log |S(i\omega)| d\omega = \pi \sum_{i=1}^{M} \mathrm{Re}(p_i)$$

(There exists a similar condition relating $T(s)$ and RHP zeros, see the lecture notes.)

G. Stein: "Conservation of dirt!"

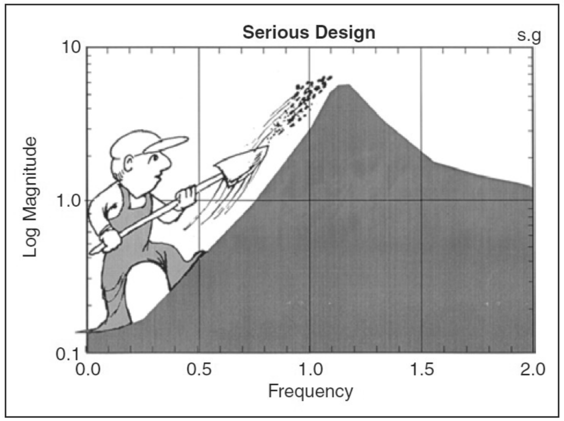


Figure 3. *Sensitivity reduction at low frequency unavoidably leads to sensitivity increase at higher frequencies.*

Picture from Gunter Stein's Bode Lecture (1985) "Respect the unstable". Reprint in *IEEE Control Systems Magazine*, Aug 2003.

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Non-minimum-phase systems

A transfer function $G(s)$ can be factored as

$$G(s) = G_{mp}(s)\, G_{nmp}(s)$$

such that

- $G_{mp}(s)$ only contains minimum-phase elements
- $G_{nmp}(s)$ contains non-minimum-phase elements and has
  - unit magnitude: $|G_{nmp}(i\omega)| = 1$
  - negative phase: $\arg G_{nmp}(i\omega) < 0$

Non-minimum-phase elements

Pole in the right half-plane at p:

$$G_{nmp}(s) = \frac{s+p}{s-p}$$

Zero in the right half-plane at z:

$$G_{nmp}(s) = \frac{z-s}{s+z}$$

Time delay of length L:

$$G_{nmp}(s) = e^{-sL}$$

Insights from loop shaping

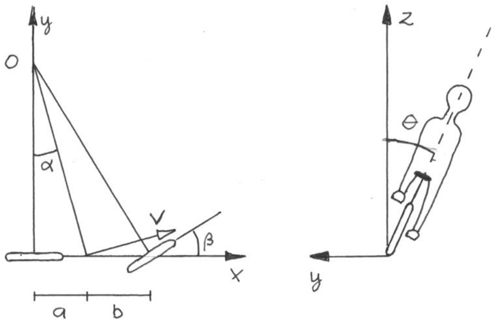
The minimum-phase part of the system can be shaped to our liking, to achieve a suitable cross-over frequency ω_c and phase margin φ_m. However,

- An RHP pole p decreases the phase by $> 90^\circ$ for $\omega < p$. To retain a reasonable phase margin, we must have $\omega_c > p$.
- An RHP zero z decreases the phase by $> 90^\circ$ for $\omega > z$. To retain a reasonable phase margin, we must have $\omega_c < z$.
- A time delay L decreases the phase by ωL. To retain a reasonable phase margin, we must have $\omega_c < \frac{\pi/2}{L} \approx \frac{1.6}{L}$.

Example: Rear-wheel steering bike

Bike example

A (linearized) torque balance can be written as



$$J\frac{d^2\theta}{dt^2} = mg\ell\theta + \frac{mV_0\ell}{b}\left(V_0\beta + a\frac{d\beta}{dt}\right)$$

Bike example, cont'd

$$J\frac{d^2\theta}{dt^2} = mg\ell\theta + \frac{mV_0\ell}{b}\left(V_0\beta + a\frac{d\beta}{dt}\right)$$

where the physical parameters have typical values as follows:

Mass:	$m = 70$ kg
Distance rear-to-center:	$a = 0.3$ m
Height over ground:	$\ell = 1.2$ m
Distance center-to-front:	$b = 0.7$ m
Moment of inertia:	$J = 120$ kgm^2
Speed (reverse sign if rear-wheel steering):	$V_0 = 5$ ms^{-1}
Acceleration of gravity:	$g = 9.81$ ms^{-2}

The transfer function from β to θ is

$$P(s) = \frac{mV_0\ell}{b}\frac{as + V_0}{Js^2 - mg\ell}$$

Bike example, cont'd

The system has an unstable pole at

$$p = \sqrt{\frac{mg\ell}{J}} \approx 2.5$$

The closed-loop system must be at least as fast as this. Moreover, the transfer function has a zero at

$$z = -\frac{V_0}{a} \approx -\frac{V_0}{0.3}$$

For the back-wheel steered bike we have the same pole but different sign of V_0 and the zero will thus be in the RHP!

An unstable pole-zero cancellation occurs for $V_0 \approx 0.75$ m/s.

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Sensitivity bounds from RHP poles/zeros

The sensitivity function must be 1 at a RHP zero z:

$$P(z) = 0 \qquad \Rightarrow \qquad S(z) := \frac{1}{1 + \underbrace{P(z)}_{0}\, C(z)} = 1$$

Similarly, the complementary sensitivity function must be 1 at an unstable pole p:

$$P(p) = \infty \qquad \Rightarrow \qquad T(p) := \frac{P(p)C(p)}{1 + P(p)C(p)} = 1$$

The Maximum Modulus principle

Suppose that all poles of the rational function $G(s)$ have negative real part. Then

$$\sup_{\omega \in \mathbb{R}} |G(i\omega)| \geq |G(s)|$$

for all s in the right half-plane.

Limits on specifications on S

THEOREM:

Given stable $W_S(s)$ and $S(s) = (1 + L(s))^{-1}$, the specification

$$\|W_S S\|_\infty \leq 1$$

can be met **only if** $|W_S(z)| \leq 1$ for every RHP zero z of $L(s)$.

Proof:

$$\|W_S S\|_\infty = \sup_{\omega \in \mathbb{R}} |W_S(i\omega)S(i\omega)| \geq |W_S(s)S(s)|$$

for all s in RHP. For $s = z$, the right hand side becomes $|W_S(z)|$, which in turn gives the necessary condition above.

Limits on specifications on T

THEOREM:

Given stable $W_T(s)$ and $T(s) = (1 + L(s))^{-1}L(s)$, the specification

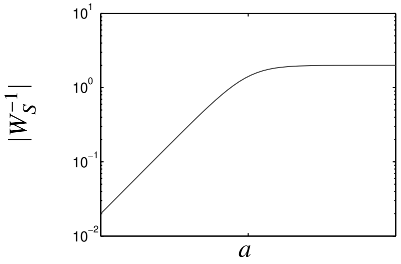
$$\|W_T T\|_\infty \leq 1$$

can be met **only if** $|W_T(p)| \leq 1$ for every RHP pole p of $L(s)$.

(Proof is analogous to the one above)

Example: Limitation from RHP zero

Assume the sensitivity specification $W_S(s) = \dfrac{s+a}{2s}, \;\; a > 0.$



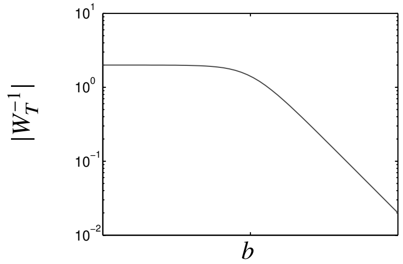
If the plant has a RHP zero in z, then $\|W_S S\|_\infty \leq 1$ is impossible to fulfill unless

$$\left| \frac{z+a}{2z} \right| \leq 1 \quad \Leftrightarrow \quad a \leq z$$

("Closed loop must be slower than z for reasonable robustness, $M_s \leq 2$")

Example: Limitation from unstable pole

Assume the compl. sensitivity specification $W_T = \dfrac{s+b}{2b}$, $b > 0$



If the plant has an unstable pole in p, then $\|W_T T\|_\infty \leq 1$ is impossible to fulfill unless

$$\left| \frac{p+b}{2b} \right| \leq 1 \quad \Leftrightarrow \quad b \geq p$$

("Closed loop must be faster than p for reasonable robustness, $M_t \leq 2$")

RHP zero and unstable pole

For a system with both RHP zero z and unstable pole p it can be shown that

$$M_s = \sup_{\omega} |S(i\omega)| \geq \left| \frac{z+p}{z-p} \right|$$

(See lecture notes for details)

If $p \approx z$ the sensitivity function must have a high peak *for every controller* C.

Example: Bicycle with rear wheel steering

$$\frac{\theta(s)}{\delta(s)} = \frac{am\ell V_0}{bJ} \cdot \frac{(-s + V_0/a)}{(s^2 - mg\ell/J)}$$

Lecture 7 – summary

- Bode's Relation and Bode's Integral
- Limitations from unstable poles, RHP zeros and time delays
  - Rules of thumb for achievable ω_c
- Limitations on specifications on S and T from unstable zeros and poles: Hard proofs using Maximum Modulus principle
- Example: Back-wheel steering bicyle – pole *and* zero i RHP